Reconciling the Investment-Uncertainty Relationship

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Abstract

Does uncertainty increase or decrease investment? Empirical evidence remains decidedly mixed, both in the time-series and in the cross-section. I present a dynamic model of investment that unifies these seemingly conflicting facts. Central to the analysis, I use a new statistic—the distance to investment—which measures how close a project is to having a positive net present value. I derive closed-form expressions for optimal investment under stochastic volatility and find three main results. (1) Over a fixed time horizon, high average uncertainty is associated with higher investment if the initial distance is "far", but lower investment if the distance is "close". (2) Uncertainty shocks increase investment only when they coincide with higher expected growth rates. (3) In general equilibrium, an uncertainty shock can raise investment during economic contractions, contrary to the partial-equilibrium intuition.

Keywords: Uncertainty, Investment, Horizon, Economic Growth

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1 Introduction

The relationship between investment and firm-level uncertainty is central to economics and finance. Uncertainty−defined by the volatility of payoffs−is a key ingredient in the firm's price of risk, which in turn drives capital budgeting and planning decisions. But despite the importance of this topic, the relationship between the two in the data is unclear.

In the time-series, a large body of work shows that uncertainty shocks depress subsequent capital investment, notably Bloom 2009, Gulen and Ion 2016, Alfaro et al. 2024 among many others. This is consistent with the "bad-news principle" of Bernanke 1983 and the classic real-options view that high volatility raises the option value of waiting. At the same time, Atanassov et al. 2024 document a *positive* impact of political uncertainty on subsequent research and development. The authors argue that their findings appeal to a time-to-build view of investment in the spirit of Bar-Ilan and Strange 1996.

Evidence in the cross-section is equally divided. Leahy and Whited 1996 and Kermani and Ma 2023 document a negative relationship between uncertainty and investment in their regressions of investment on firm-level risk, arguing in favor of models of irreversible investment (Dixit and Pindyck 1994). On the other hand, Chang et al. 2024 documents a positive relationship in support of their model and Abel 1983. Kang 2025 even finds that the relationship can change depending on the horizon at which uncertainty is measured.

These conflicting facts raise fundamental questions about the underlying economic mechanism. One natural interpretation is that the sign depends on measurement choices: quarterly versus annual investment rates, equity versus asset volatility, physical versus intangible capital or differences in econometric specification. But it is unclear how these choices matter theoretically, nor is there a unifying framework that can rationalize both a positive and negative relationship; existing models take a stance on a single sign.

This paper proposes a unified framework that reconciles these conflicting empirical findings. I show that a single economic mechanism, the distance to investment, organizes both cross-sectional and time-series variation in investment responses to uncertainty. The distance to investment is simply a statistic that measures how close a project is to having a positive net present value; it determines whether a firm is "far" or "close" from exercising an option to invest. The role of volatility is straightforward: it affects the likelihood at which an unprofitable project becomes profitable and vice-versa.

Intuitively, suppose, for example, the current price of output is \$1.50 but investment becomes profitable only once the price reaches \$2. When the price lies below the threshold, the distance to investment is large. In this region, increased volatility is beneficial: large price movements accelerate the probability of crossing the threshold, raising investment over any fixed horizon. In contrast, if the current price is \$2.50—above the threshold—and the drift of prices is positive, then high volatility increases the chance of falling below it. Here, volatility hurts investment when projects are nearly or already profitable. This single statistics delineates the conditions under which uncertainty encourages or discourages investment. In the cross-section, firms that face a larger share of projects that are initially unprofitable gain more from volatility, whereas firms with projects close to profitability do not.

To formalize the mechanism, I develop a tractable dynamic model of irreversible investment (Section 2.1) in which new investment projects arrive over time. Each project consists of a bundle of factor inputs, incurs a fixed cost, and expires if not immediately adopted. With this set-up, the firm solves a sequence of optimal stopping problems. I derive closed-form solutions for the expected occupation time above the investment thresholds using tools in complex analysis, a key statistic that govern investment rates across horizons.

To reconcile the conflicting signs in the time-series, I then incorporate stochastic volatility in Section 2.3 using a two-state Markov regime-switching process, in which short bursts of high uncertainty is followed by prolonged periods of low uncertainty. I show that when the firm finds itself in the high uncertainty regime, its optimal investment decision includes a precautionary motive that increases distance. Because investment is irreversible and most time is spent in the low-volatility regime, the firm delays investment during high-uncertainty periods to avoid an unfavorable change to prices before conditions normalize. Consequently, uncertainty shocks typically depress investment. The only exception occurs when when projects require time to become profitable and uncertainty shocks are accompanied by increases in expected growth rates.

In Section 2.4, I endogenize the firm's cost of capital within a general-equilibrium framework, an important departure from the standard approach that assumes a constant discount rate. This is important since temporary shocks to aggregate uncertainty should also be accompanied by changes to the market price of risk; in contrast, most of the literature uses a constant discount rate. Importantly, the investment threshold for a given firm is a function of the equilibrium risk-free rate, which in turn is determined by expectations about future aggregate investment. One key implication is that the impact of uncertainty shocks varies systematically with the state of the economy. During economic slowdowns, an uncertainty shock can stimulate investment because the riskless rate is relatively low, thereby keeping investment thresholds low. As a result, high volatility may help firms reach their investment triggers sooner. In contrast, when the economy is overheating, an uncertainty shock compounded by a higher riskless rate raises investment thresholds too far, reducing investment.

Finally in Section 3, I measure the distance to investment empirically and find preliminary evidence for the model prediction in the cross-section. In particular, I regress physical capital investment on firm-level uncertainty, a proxy for distance, and an interaction term of the two. Consistent with the model, I find that the coefficient on the interaction term is positive while the coefficients on uncertainty and distance are negative. This supports the model prediction in the cross-section that firms who naturally face a larger proportion of investment opportunities that are not immediately profitable benefit from higher uncertainty.

Together, these results provide a unified explanation for the conflicting empirical findings on uncertainty and investment. The distance to investment emerges as a useful statistic that organizes both time-series and cross-sectional variation, delivering clear conditions under which uncertainty stimulates or depresses investment.

1.1 Related Literature

The primary contribution of this paper is to provide a tractable framework that can explain the different signs in both the time-series and in the cross-section.

In particular, the literature on investment under uncertainty with constant volatility can be divided into models that predict a positive cross-sectional relationship between firm-level uncertainty and investment and ones that predict a negative relationship. In the former camp, the convexity of payoffs with respect to the stochastic variable leads to a positive relationship, starting from Oi 1961, Hartman 1972 and Abel 1983, and as recently as Chang et al. 2024. Other models that predict a positive relationship without relying on convexity include Roberts and Weitzman 1981 and Bar-Ilan and Strange 1996 who study sequential and multi-stage projects.[1] In the latter camp, non-convex investment costs lead to an investment "no-action" region. In the comparative statics, because an increase in σ expands the no-action region, authors then conclude that uncertainty delays investment (Bernanke 1983; Pindyck 1988; Dixit and Pindyck 1994).

My paper contributes to this literature by rationalizing both sides using a single statistic: the distance to investment, without changing the model assumptions along the way. To my knowledge, Caballero 1991 also reconciles the conflicting signs in the cross-section, though the author requires perfect competition and symmetric adjustment costs for a positive sign, and imperfect competition and asymmetric adjustment costs for a negative one.

On the literature on investment under uncertainty with stochastic volatility, this paper is potentially the first to present a dynamic model that can rationalize an *increase* in investment in response to an uncertainty shock.

The closest paper to mine is likely Bloom 2009. One of the key differences between our papers is that in Bloom 2009, the author assumes that firms are close to the optimal exercise boundary before an uncertainty shock. Mechanically, an uncertainty shock will expand the

[1] In Roberts and Weitzman 1981, the firm has an option to abandon a project before completion. In their model, the firm learns about the terminal payoff progressively after completing a stage of the project. When uncertainty is high, completing early stages is valuable due to the high value of information gained. In this manner, high uncertainty can "increase" investment. But because the focus of their work is on the optimal exploration and development for different type of processes, it is unclear if their results translate to average effects over multiple projects.

no-action region, which leads to an immediate delay in investment before catching up due to the drift of stochastic process. In contrast, the distance to investment statistic is defined by how far the optimal exercise boundary is from the current level of the state variable. Hence, explaining the finding that an uncertainty shock can increase investment comes from this difference.

Moreover, my paper endogenizes the firm's cost of capital in general equilibrium whereas the cost of capital in Bloom 2009 is a constant, regardless of the volatility regime. In fact, this omission typically holds for most papers in this literature. Consequently, I show that a decrease in investment in response to an uncertainty shock occurs when the economy is growing; on the other hand, when the economy is contracting, my model predicts that an uncertainty shock can boost investment.

2 Model

I first outline the setup of the model in Section 2.1. To build intuition, in Section 2.2, I derive the firm's optimal investment policy subject to an exogenous stochastic discount factor in the economy in which the firm faces a constant level of uncertainty. In Section 2.3, I add stochastic volatility to this framework. Finally, in Section 2.4, I derive the general equilibrium results.

2.1 The Setup

In this section, I derive the firm's optimal investment policy subject to an exogenous stochastic discount factor in the economy. I first focus on the case in which the firm faces a constant level of uncertainty, then add stochastic volatility into the framework.

2.1.1 Technology

Time is continuous. Consider a competitive firm that produces consumption goods in each period with a constant returns to scale and constant elasticity of substitution (CES) production technology,

$$y(t, A) = \Big[\alpha(z_t)^{\varepsilon} + (1-\alpha)(k_t)^{\varepsilon}\Big]^{\frac{1}{\varepsilon}} A_t^{\theta} \tag{1}$$

where A_t is a stochastic process that drives the productivity of output,

$$\mathbf{d}A_t = A_t \mu \mathbf{d}t + A_t \sigma \mathbf{d}B_t$$

and B_t is a standard Brownian motion. In equation (1), the parameter α controls the relative share while ε relates to the elasticity of substitution between the two production inputs (complements at $\varepsilon = 0$, substitutes at $\varepsilon = 1$). The parameter θ controls the curvature of output with respect to the stochastic variable.[2] Throughout this paper, I consider the linear case, $\theta = 1$.

Importantly, z_t and k_t denote two types of capital inputs allocated to the production of the good. For simplicity, I assume production does not require labor. Moreover, to facilitate the derivation of closed-form expressions, I make the following assumption regarding capital accumulation in this model,

Assumption 1. An investment project is a bundle of inputs $(\mathfrak{z}_t, \mathfrak{k}_t)$. The firm can acquire this bundle at a fixed cost q. Once adopted, production expands from $[\alpha(z_t)^{\varepsilon} + (1-\alpha)(k_t)^{\varepsilon}]^{\frac{1}{\varepsilon}}$ to $[\alpha(z_t + \mathfrak{z}_t)^{\varepsilon} + (1-\alpha)(k_t + \mathfrak{k}_t)^{\varepsilon}]^{\frac{1}{\varepsilon}}$.

Assumption 2. With probability 1, an investment project arrives in each unit of time. The *composition* of the investment projects are independent random draws from uniform distributions on $[\underline{\mathfrak{z}}, \bar{\mathfrak{z}}]$ and $[\underline{\mathfrak{k}}, \bar{\mathfrak{k}}]$ at the time of arrival. Moreover, a project expires if it is not adopted by the arrival of the next project.

Given the cost q, Assumption 1 simplifies the investment choice to an optimal stopping problem. Importantly, with two capital inputs and the CES aggregator, the marginal benefit of an investment project to the firm differs on a project-by-project basis. When the composition of the investment project closely mirrors the firm's existing capital stock, the project is not very valuable. As a result, due to Assumption 2, the firm may skip projects that do not meet its current needs. This conflict highlights the fact that a competitive firm may not always have access to its preferred type of capital; some projects are more beneficial than others.

2.2 Optimal Investment

In order to describe the firm's optimal investment policy, I make the following assumptions. Suppose markets are complete, Modigliani-Miller theorem holds, and there exists an exogenous stochastic discount factor in the economy, $\mathbf{d}M_t/M_t = -r\mathbf{d}t - \kappa\mathbf{d}B_t$.

Since projects instantly expire, the decision rule is to accept projects with a positive net present value (NPV). First proceeding heuristically, suppose the firm only had access to a single investment project that will expire if not adopted today. Under the risk-neutral

[2] In many models with Cobb-Douglas production (such as Abel 1983), the adjustability of labor input leads indirect utility to be a convex function of the stochastic variable. Here, I assume that labor is not required in the production of the good. For the purposes of comparison, the parameter θ allows me to incorporate convexity into the model.

measure, the decision rule is to invest if

$$\left[\frac{\left[\alpha(z_t+\mathfrak{z}_t)^{\varepsilon}+(1-\alpha)(k_t+\mathfrak{k}_t)^{\varepsilon}\right]^{\frac{1}{\varepsilon}}}{r-\mu+\kappa\sigma}-\frac{\left[\alpha(z_t)^{\varepsilon}+(1-\alpha)(k_t)^{\varepsilon}\right]^{\frac{1}{\varepsilon}}}{r-\mu+\kappa\sigma}\right]\times A_t^{\theta}-q\geq 0$$

The terms inside the brackets correspond to the incremental output gained from investing in this project, and this value follows the Gordon-growth formula, assuming $r-\mu+\kappa\sigma>0$.

More formally, set $\theta=1$ and apply a change of variable from A_t to $x_t=\log A_t$. Moreover, for ease of notation, let $\zeta_n \doteq [\alpha(z_0+\sum_n \mathfrak{z}_n)^{\varepsilon}+(1-\alpha)(k_0+\sum_n \mathfrak{k}_n)^{\varepsilon}]^{1/\varepsilon}$. The subscript n indicates the number of projects that are completed, where z_0 and k_0 are the initial capital stock and $\varkappa_n$ and $\mathfrak{k}_n$ are the nth realizations of the project draws. Finally, let $\tilde{\zeta}_{n+1}$ denote the case in which the composition of the $n+1$th project (which will arrive in the future) uses expected values of the random draws. The Hamilton-Jacobi-Equation (HJB) equation can be written:

$$rV(\zeta_n,x)=\zeta_n e^x+(\mu-\kappa\sigma-0.5\sigma^2)V_x(\zeta_n,x)+\frac{1}{2}\sigma^2 V_{xx}(\zeta_n,x)+\left[V(\tilde{\zeta}_{n+1},x)-V(\zeta_n,x)-q\right]$$

with boundary conditions:

$$V(\zeta_n,\widehat{x})=V(\zeta_{n+1},\widehat{x})-q$$

where $\widehat{x}$ denotes the threshold value at which the firm adopts today's project. $V(\zeta_0,x)$ is given by

$$V(\zeta_n,x)=\left[\frac{\zeta_n}{1+\delta}+\frac{1}{1+\delta}\frac{\tilde{\zeta}_{n+1}}{\delta}\right]\times e^x-\frac{q}{1+\delta}$$

Moreover, $\delta \doteq r-\mu+\kappa\sigma$ and $\tilde{\zeta}_1$ highlights the fact that the composition of tomorrow's project is unknown. Applying the boundary condition, which effectively gives a zero NPV condition,

$$\left[\frac{\zeta_n}{1+\delta}+\frac{1}{1+\delta}\frac{\tilde{\zeta}_{n+1}}{\delta}\right]\times e^{\widehat{x}}=\left[\frac{\zeta_{n+1}}{1+\delta}+\frac{1}{1+\delta}\frac{\tilde{\zeta}_{n+2}}{\delta}\right]\times e^{\widehat{x}}-q$$

we can identify the threshold level of x such that the NPV is positive,

$$\widehat{x}=\log\left(\frac{\varphi^+}{\varphi^+-1}\frac{q}{\frac{1}{1+\delta}\left[\zeta_{n+1}-\zeta_n+\frac{\tilde{\zeta}_{n+2}}{\delta}-\frac{\tilde{\zeta}_{n+1}}{\delta}\right]}\right)$$

where $\varphi\pm=-(\mu-.5(\sigma^i)^2)/(\sigma^i)^2\pm\sqrt{2(1+r)(\sigma^i)^2+(\mu-.5(\sigma^i)^2)^2}/(\sigma^i)^2$ The above equation has an intuitive interpretation. The threshold level at which the firm invests is a ratio of the investment cost over the incremental gain in output from this project. The greater the marginal benefit, the lower the required threshold and the more likely the firm adopts it. In the denominator, the difference $\zeta_1 - \zeta_0$ reflects the immediate gain to production while the difference in the next two terms, $\frac{\tilde{\zeta}_{n+2}}{\delta} - \frac{\tilde{\zeta}_{n+1}}{\delta}$, reflects the incremental gain from future projects if today's project is adopted.

To transition from investment thresholds to the firm's investment *rate*, we can use the fact that if one project arrives at each point in time, the expected time the stochastic variable x_t spends outside of the investment region gives a natural characterization of the expected number of projects the firm adopts over a fixed time interval $[t, T]$. Define the expected occupation time,

$$f(t,x) = \mathbb{E}_t\left[\int_t^T \mathbb{1}_{x_s \geq \widehat{x}_s}\right] \mathbf{d}s \tag{2}$$

which allows for the fact that $\widehat{x}_t$ can change as the firm adopts more projects. Then, the investment rate is given by $I_T = f(t,x)/T$. The following lemma characterizes the investment rate to this setting.

Lemma 1. Fix T. Suppose $z_t/k_t = \tilde{\mathfrak{z}}/\tilde{\mathfrak{k}}$. Furthermore, let $\bar{\mu} \doteq \mu - 0.5\sigma^2$. Then,

$$I_T = \frac{1}{T}\int_t^T \left[1 - \Phi\left(\frac{\widehat{x} - x_0 - \bar{\mu}s}{\sigma\sqrt{s}}\right)\right] \mathbf{d}s \tag{3}$$

where x_0 is the initial position of x.

Lemma 1 uses the well-known formula for the density that $\Pr(x_s > \widehat{x})$ by time s. Moreover, if the composition of the firm's existing assets mirrors the average composition of investment projects, due to the homogeneity of the production function, $\widehat{x}$ is a constant in expectation even as the firm adopts more and more projects.

Though the formula in Equation 3 can be evaluated numerically without issue, we can derive more explicit representations by rewriting the expectations operator in Equation 2 in terms of a differential equation.

First, appealing to the Feynman-Kac formula, we can represent the conditional expectation $f(t,x)$ as a solution to the forward equation,

Lemma 2. $f(t,x)$ obeys the following partial differential equation:

$$\frac{\partial f(t,x)}{\partial t} + \left[\mu - \frac{1}{2}\sigma^2\right]\frac{\partial f(t,x)}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f(t,x)}{\partial x^2} = -\mathbb{1}_{x>\widehat{x}}$$

with the terminal condition $f(T,x) = 0$.

Proof. See Appendix 5.1. $\square$

As always, it is convenient to convert the PDE into an ODE. Consider the change of variable $\tau = T - t$ and the Laplace transform, $\overline{f}(\xi, x) = \mathscr{L}\{f\} \doteq \int_0^\infty e^{-\xi\tau} f(\tau, x)\mathbf{d}t$. [3]

The PDE then becomes:

$$-\xi\overline{f}(\xi, x) + \left[\mu - \frac{1}{2}\sigma^2\right]\frac{\partial \overline{f}(\xi, x)}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 \overline{f}(\xi, x)}{\partial x^2} = -\frac{1}{\xi}\mathbb{1}_{x>\widehat{x}}$$

$$f(\tau, x) = \mathscr{L}^{-1}\{\overline{f}(\xi, x)\} = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\xi\tau}\overline{f}(\xi, x)\mathbf{d}\xi \tag{4}$$

Proposition 1. Let $\bar{\mu} \doteq \mu - 0.5\sigma^2$. The expected occupation time over the time interval $\tau \doteq T - t$ is

$$\begin{aligned} f(\tau, x) = \tau - \frac{\widehat{x} - x}{\bar{\mu}} - \frac{1}{2}\frac{\sigma^2}{\bar{\mu}^2} + \frac{1}{2\pi}\int_0^\infty \frac{1}{\xi^2} e^{(-\frac{1}{2}\frac{\bar{\mu}^2}{\sigma^2} - \xi)\tau - \frac{2\bar{\mu}}{\sigma^2}(x - \widehat{x})} \\ \times \left[\sin\left(\frac{2\sqrt{2\xi}}{\sigma}(x - \widehat{x})\right) - \frac{\bar{\mu}}{\sigma\sqrt{2\xi}}\cos\left(\frac{2\sqrt{2\xi}}{\sigma}(x - \widehat{x})\right)\right]\mathbf{d}\xi \end{aligned} \tag{5}$$

Numerically, the terms inside the integrand in Equation (5) is relatively small si we can largely focus on the terms: $\tau - \frac{\widehat{x} - x}{\bar{\mu}} - \frac{1}{2}\frac{\sigma^2}{\bar{\mu}^2}$. In contrast to Equation (3), the above proposition gives us a cleaner mapping between the parameters and the expected occupation time.

Furthermore,

Corollary 1. Asymptotically as $\tau \to 0$,

$$f(\tau, x) \sim \tau - \frac{\widehat{x} - x}{\bar{\mu}} - \frac{1}{2}\frac{\sigma^2}{\bar{\mu}^2} + \frac{1}{\sqrt{\pi}} e^{\xi_b\tau - \frac{2\bar{\mu}}{\sigma^2}(x - \widehat{x})}\left[\left(\frac{3\psi}{4\xi_b^2} + \frac{1}{\xi_b^3\psi}\right)\tau^{-3/2} - \frac{1}{\xi_b^2\psi}\tau^{-1/2}\right] \tag{6}$$

where $\psi \doteq \frac{2\sigma}{\sqrt{\bar{\mu}}}(x - \widehat{x})$.

Using the above results, Figure 1 plots investment rates for different values of σ. Importantly, the initial point x_0 is set equal to zero while the investment threshold $\widehat{x}$ takes values

[3] The Laplace transform has the following properties:

$$\mathscr{L}\left\{\frac{\partial f}{\partial t}\right\} = \int_0^\infty e^{-\xi\tau}\frac{\partial f}{\partial t}\mathbf{d}t = -\xi\overline{f}(\xi, x) + f(0, x)$$

via the chain rule and integration by parts and

$$\mathscr{L}\left\{\frac{\partial f}{\partial x}\right\} = \int_0^\infty e^{-\xi\tau}\frac{\partial f}{\partial x}\mathbf{d}t = \frac{\partial}{\partial x}\int_0^\infty e^{-\xi\tau} f\mathbf{d}t = \frac{\partial \overline{f}}{\partial x}$$

via Leibniz's Rule. While Fourier Transforms have been used before in Finance, because the transform variable in this setting is time and not probability, the Laplace transform−which only integrates over the positive real line−is the appropriate transformation.

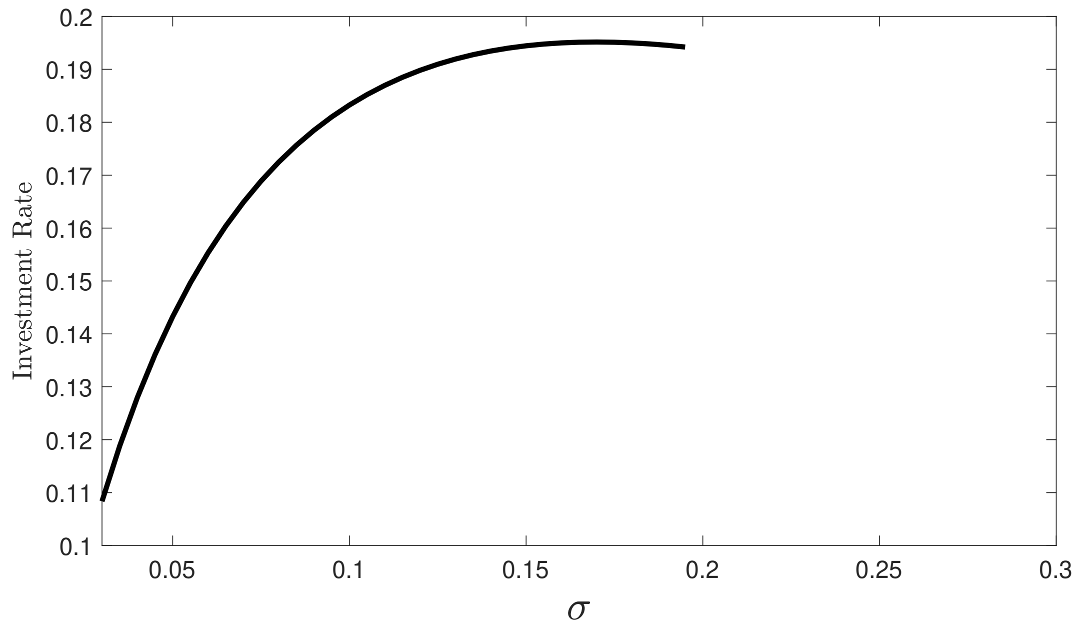


Figure 1: **Investment rate over 5 years.** $\mu = 0.02, r = 0.05, \kappa = 0.3$. The initial point x_0 while is set to 0 while the investment threshold $\widehat{x} \sim 0.25$ such that the distance is initially "far".

greater than 0.2, implying that projects are initially unprofitable. For the values of σ such that $\bar{\mu} \doteq \mu - 0.5\sigma^2 > 0$, an increase in σ locally increases the investment rate until with too much variation, the distribution of occupation times becomes "defunct" ($\bar{\mu} < 0$) and we can no longer compute an expected value.

However, I show in Figure 2 that the positive relationship between investment and uncertainty strongly depends on the initial distance between x_0 and $\widehat{x}$. In this figure, I again plot the firm's expected investment rate, this time varying the initial point x_0. For various levels of σ, the optimal investment threshold $\widehat{x}$ is between $0.2 - 0.3$. Thus, the figure plots situations in which the firm is already above the action region, $x_0 > \widehat{x}$, i.e., there is no delay and the firm immediately invests. In the left plot, the drift μ is set to a constant value while on the right, μ is increasing in σ to account for the possibility that higher volatility is compensated by higher growth rates.

Importantly, if the initial distance to the investment threshold is "far" (values of x_0 close to zero), high uncertainty is associated with higher investment. The reason is that high volatility increases the likelihood of low probability events. As a result, the firm is likely to reach the investment threshold sooner and adopt a larger fraction of investment projects over a given time interval. On the other hand, if the firm is initially close to or is already above the threshold, we see that high uncertainty is associated with lower investment. This is due to the fact that when the firm is already within the action region, high volatility introduces

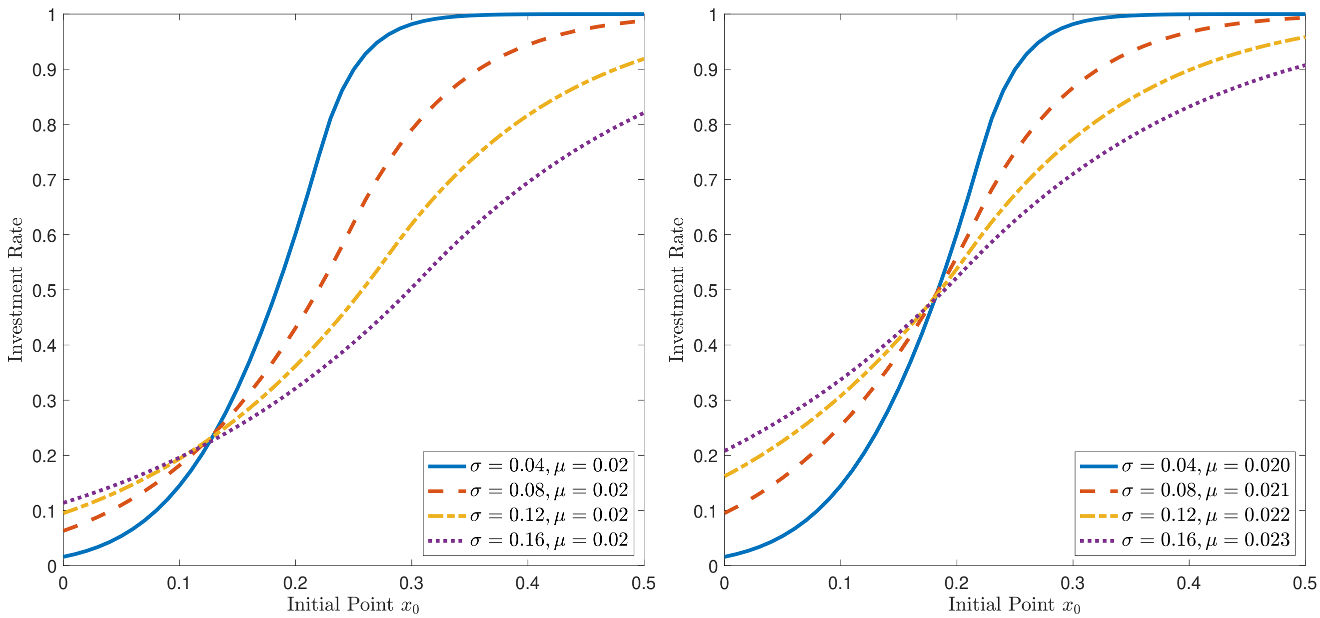


Figure 2: **Investment rate for different initial points x_0 over $T = 5$ years.** In the left panel, the optimal investment threshold $\widehat{x}$ for each level of σ is: $0.22,\ 0.25,\ 0.28,\ 0.31$ and for the right panel: $0.22,\ 0.22,\ 0.21,\ 0.21$.

a probability that the firm falls out of it. The distance to investment $x_0 - \widehat{x}$ therefore plays an important role in understanding the investment-uncertainty relationship.

With a final caveat, since the investment rate is also a function of the horizon T, Figure 3 plots the investment rate for different values of T while fixing $x_0 = 0$. On the left panel, μ is held fixed at $\mu = 0.02$ while on the right, μ is increasing in σ. Though the initial distance is far, the figure shows that extending the time interval at which we measure investment can reverse the investment-uncertainty relationship despite the preceding two figures. In this case, because $\mu > 0$, the stochastic variable x_t will eventually reach $\widehat{x}$. By considering a long time interval, a higher value of σ needlessly increases the likelihood that projects fall out of profitability in the long-run. Hence, not only does the initial distance to investment matter, but so does the econometrician's choice of time interval.

Indeed, Chang et al. 2024 finds a positive coefficient in regressions of physical capital investment on average uncertainty using quarterly data, whereas Leahy and Whited 1996 and Kermani and Ma 2023, who run the same regressions, find a negative coefficient using annual data. Using the tools studied in this section$-$the distance to investment and the choice of $T-$, we can rationalize these conflicting signs and explain why a positive relationship over short time intervals may turn to a negative relationship over longer ones.

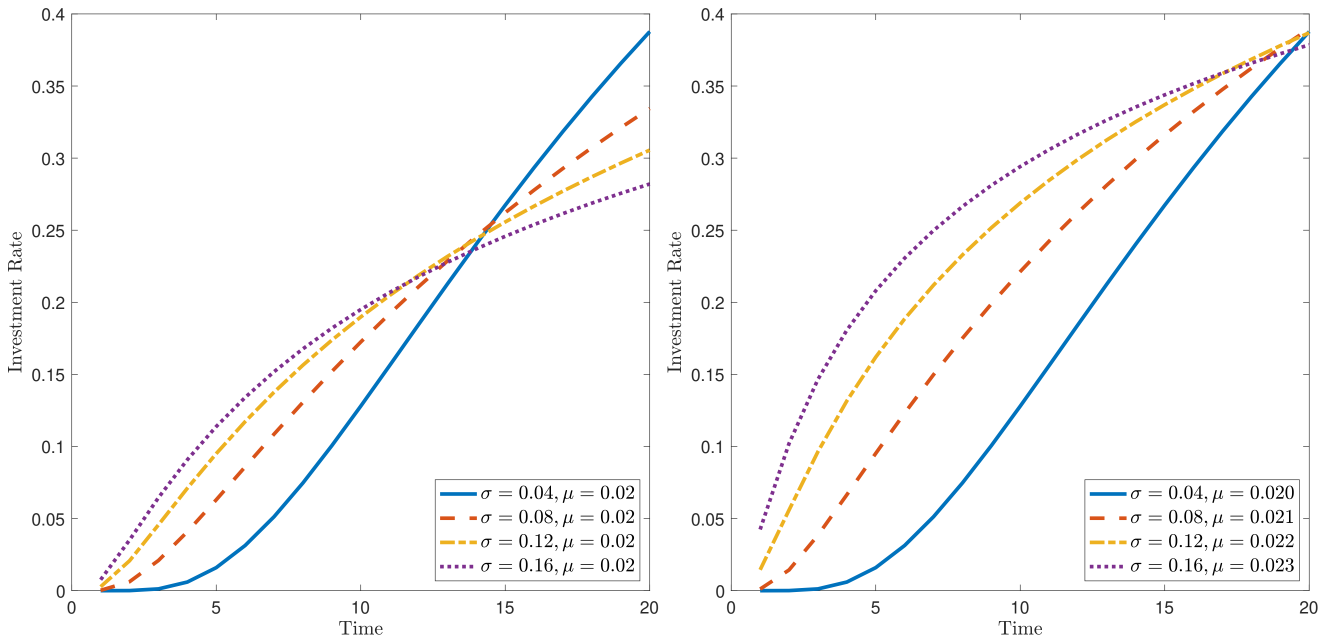


Figure 3: **Investment rate over different horizons T.**

2.3 Stochastic Volatility

The previous section focused on differences in the rate of investment if a firm faced different levels of constant volatility. In this section, I introduce stochastic volatility to study how the same firm responds to changes in uncertainty, separating a time-series effect from the cross-sectional effect of uncertainty. Now, suppose the economy alternates between two volatility regimes. I use $\eta^L \mathbf{d}t, \eta^H \mathbf{d}t$ to denote the risk-neutral probabilities of a transition. A H volatility regime can only be followed by a L regime and vice-versa. The system of Hamilton-Jacobi-Bellman equations can be expressed (in terms of A),

$$\begin{aligned} rV^H(z,k,A) &= \zeta_0 A + (\mu - \kappa\sigma^H) A \frac{\partial V(z,k,A)}{\partial A} + \frac{1}{2}(\sigma^H)^2 A^2 \frac{\partial^2 V(z,k,A)}{\partial A^2} \\ &\quad + \left[V^H(z+\tilde{\mathfrak{z}}, k+\tilde{\mathfrak{k}}, A) - V^H(z,k,A) - q\right] + \eta^L \left[V^L(z,k,A) - V^H(z,k,A)\right] \\ rV^L(z,k,A) &= \zeta_0 A + (\mu - \kappa\sigma^L) A \frac{\partial V(z,k,A)}{\partial A} + \frac{1}{2}(\sigma^L)^2 A^2 \frac{\partial^2 V(z,k,A)}{\partial A^2} \\ &\quad + \left[V^L(z+\tilde{\mathfrak{z}}, k+\tilde{\mathfrak{k}}, A) - V^L(z,k,A) - q\right] + \eta^H \left[V^H(z,k,A) - V^L(z,k,A)\right] \end{aligned}$$

with boundary conditions

$$\begin{aligned} V^H(z,k,A^{*H}) &= V^H(z+\mathfrak{z}, k+\mathfrak{k}, A^{*H}) - q \\ V^L(z,k,A^{*H}) &= V^L(z+\mathfrak{z}, k+\mathfrak{k}, A^{*H}) - q \end{aligned}$$

Applying the change of variable $x = \log A$ and defining the following:

$$\nu^H \doteq V^H_x \quad ; \quad \nu^L \doteq V^L_x$$

The ODE can be written as:

$$\nabla \begin{bmatrix} V^H \\ \nu^H \\ V^L \\ \nu^L \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{(1+r+\eta^L)}{.5(\sigma^H)^2} & -\frac{\mu-\kappa\sigma^H-.5(\sigma^H)^2}{.5(\sigma^H)^2} & -\frac{\eta^L}{.5(\sigma^H)^2} & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\eta^H}{.5(\sigma^L)^2} & 0 & \frac{(1+r+\eta^H)}{.5(\sigma^L)^2} & -\frac{\mu-\kappa\sigma^L-.5(\sigma^L)^2}{.5(\sigma^L)^2} \end{bmatrix}}_{\doteq M} \begin{bmatrix} V^H \\ \nu^H \\ V^L \\ \nu^L \end{bmatrix} + \begin{bmatrix} 0 \\ \left[\zeta_0 + \frac{\tilde{\zeta}_1}{\sigma^H}\right] e^s - q \\ 0 \\ \left[\zeta_0 + \frac{\tilde{\zeta}_1}{\sigma^L}\right] e^s - q \end{bmatrix} \tag{7}$$

In Appendix 5.3, I show that there are constants a_1, a_3, b_1, b_3, eigenvalues ω_1, ω_3 and elements of eigenvectors $\mathbf{v}^{(1)}, \mathbf{v}^{(3)}$ corresponding to these eigenvalues such that,

Proposition 2. In volatility regime H, investment occurs the first time

$$\tau^{H*} \doteq \inf \left\{ t : x_t \geq \log \left(\frac{\omega^{(1)}}{\omega^{(1)} - 1} \frac{v_1^{(3)} - v_3^{(3)}}{v_1^{(3)}(b_3 - a_3) - v_3^{(3)}(b_1 - a_1)} \times q \right) = \widehat{x}^H \right\} \tag{8}$$

In volatility regime L, investment occurs the first time

$$\tau^{L*} \doteq \inf \left\{ t : x_t \geq \log \left(\frac{\omega^{(3)}}{\omega^{(3)} - 1} \frac{v_1^{(1)} - v_3^{(1)}}{v_1^{(1)}(b_3 - a_3) - v_3^{(1)}(b_1 - a_1)} \times q \right) = \widehat{x}^L \right\} \tag{9}$$

Proof. See Appendix 5.3. □

Again, the optimal investment threshold is a ratio of the fixed investment cost divided by the marginal benefit to output of investing in this project, scaled by some constant. This time, the marginal benefit is a weighted average of two terms, splitting it into the expected marginal benefit at each regime.

Proposition 2 indicates that there are two investment thresholds ($\widehat{x}^H$ and $\widehat{x}^L$) corresponding to each regime. As a result, in order to compute an investment rate as before, it is necessary to account for the possibility of a regime change in the future.

More formally, let $f^H(t, x) \doteq \mathbb{E}_t \left[\int_t^T \mathbb{1}_{x_s \geq \widehat{x}^H} \mathbf{d}s \right]$ and $f^L(t, x) \doteq \mathbb{E}_t \left[\int_t^T \mathbb{1}_{x_s \geq \widehat{x}^L} \mathbf{d}s \right]$ denote the expected occupation time above the investment thresholds $\{\widehat{x}^H, \widehat{x}^L\}$. We can represent these conditional expectations as solutions to the equations

Lemma 3. $f^H(t,x)$ and $f^L(t,x)$ obey the following system of partial differential equations:

$$\frac{\partial f^H(t,x)}{\partial t} + \left[\mu^H - \frac{1}{2}(\sigma^H)^2\right]\frac{\partial f^H(t,x)}{\partial x} + \frac{1}{2}(\sigma^H)^2\frac{\partial^2 f^H(t,x)}{\partial x^2} + \eta^L\Big[f^L(t,x) - f^H(t,x)\Big] = -\mathbb{1}_{x>\widehat{x}^H}$$
$$\frac{\partial f^L(t,x)}{\partial t} + \left[\mu^L - \frac{1}{2}(\sigma^L)^2\right]\frac{\partial f^L(t,x)}{\partial x} + \frac{1}{2}(\sigma^L)^2\frac{\partial^2 f^L(t,x)}{\partial x^2} + \eta^H\Big[f^H(t,x) - f^L(t,x)\Big] = -\mathbb{1}_{x>\widehat{x}^L}$$

with the terminal conditions $f^H(T,x) = 0$ and $f^L(T,x) = 0$.

Proof. See Appendix 5.4. □

Like before in Section 2.2, we proceed by reducing the above system of PDEs into a system of ODEs via the Laplace transform, then invert the solution to the system of ODEs to get back the expected occupation time from t to T.

Proposition 3. The expected investment rate in each regime is given by

$$I_T^H = \frac{1}{T-t}\sum_j^2 \operatorname{Res}\left(e^{\xi t}\overline{f}^H(\xi,x);\xi_j\right)$$
$$I_T^L = \frac{1}{T-t}\sum_j^2 \operatorname{Res}\left(e^{\xi t}\overline{f}^L(\xi,x);\xi_j\right)$$

where $\xi_1 = 0$ and $\xi_2 = \eta^L + \eta^H$. The notation $\operatorname{Res}(\cdot)$ indicates the residue of the function at the jth order pole ξ_j.[4]

Proof. See Appendix 5.3. □

Using the above results, Figure 4 plots the expected investment rate in each volatility regime as a function of the initial point x_0. Importantly, setting $\mu^H = \mu^L$, the figure shows that the firm has a lower investment rate in the higher volatility regime, regardless of the

[4] The definition of residues and poles is first presented in Appendix 5.2. To show the usefulness of the residue theorem, consider the integral

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\mathbf{d}x = \pi$$

over the function $f(x) = \frac{1}{x^2+1}$. This function has two simple poles: i and $-i$. Along the complex plane, note that the integral $\int_{-\infty}^{\infty} f(x)\mathbf{d}x$ is a straight horizontal line along the real axis. To use contour integration (and hence, the residue theorem), we can close the contour arbitrarily along the upper-half plane, enclosing i. Then, by the residue theorem,

$$\oint_\Omega f(x)\mathbf{d}x = 2\pi i \operatorname{Res}(f;i) = 2\pi i \times \lim_{x\to i}\,(x-i)\frac{1}{x^2+1} = 2\pi i \times \lim_{x\to i}\frac{1}{x+i} = 2\pi i \times \frac{1}{2i} = \pi$$

distance to investment. Hence, in most cases, an uncertainty shock leads to a decrease in investment.

Of course, the mechanism with stochastic volatility differs than in the static setting. The argument is as follows. Suppose the firm is in the high regime. With the specification that $\eta^L > \eta^H$, one risk of investing too early is that due to the high uncertainty, the level of the stochastic variable may reach an undesirable level before the arrival of the low regime, in which the firm is likely to be stuck around that range. Hence, the firm is reluctant to invest while in the high regime. We can see this more clearly in Figure 6. As η^L increases, as it becomes more likely that the low volatility regime will arrive, there is less risk of a sudden, large shock that makes the firm regret investing early. By shortening the expected duration of the high uncertainty regime, the firm is more willing to invest in the high regime.

Figure 5 then studies in setting in which μ^L and μ^H may differ. Importantly, the only scenario in which an uncertainty shock leads to an increase in investment is when $\mu^H > \mu^L$ and the initial distance is far. We can see this in the bottom-most plot. Furthermore, we can see that $\widehat{x}^H < \widehat{x}^L$ as well, indicating that a higher growth rate μ^H in the high volatility regime mitigates the precautionary motive. When the initial distance is far, high volatility again enables the firm to reach the target threshold sooner.

This result is consistent with Atanassov et al. 2024 who find an increase in investment in response to uncertainty shocks. In their paper, the authors focus on R&D investment and argue that the effect of uncertainty may differ for R&D, in which projects typically take much longer to mature and turn profitable. Moreover, due to the nature of R&D, it is plausible that higher variance may be accompanied by a higher mean.

To summarize the results in Sections 2.2 and 2.3, I first studied the case in which the firm faces a constant level of uncertainty. By varying the value of σ, I showed that whether investment has a positive or negative relationship with uncertainty depends on the initial distance before which a project turns profitable. If the distance is far, high volatility increases the likelihood that the project turns profitable sooner while if the distance is close, higher volatility increase the likelihood that project becomes unprofitable.

In the time-series, firms typically reduce investment in response to uncertainty shocks. This is the standard result in the literature and in my paper, comes from a precautionary motive. On the other hand, I show that an exception occurs when uncertainty shocks are accompanied by increases in expected growth rates and at the same time, the initial distance is far.

In this partial-equilibrium setting, I provide a unified explanation for the mixed empirical findings on uncertainty and investment. The distance to investment is a simple statistic that organizes both time-series and cross-sectional variation, delivering clear conditions under which uncertainty increases or reduces investment.

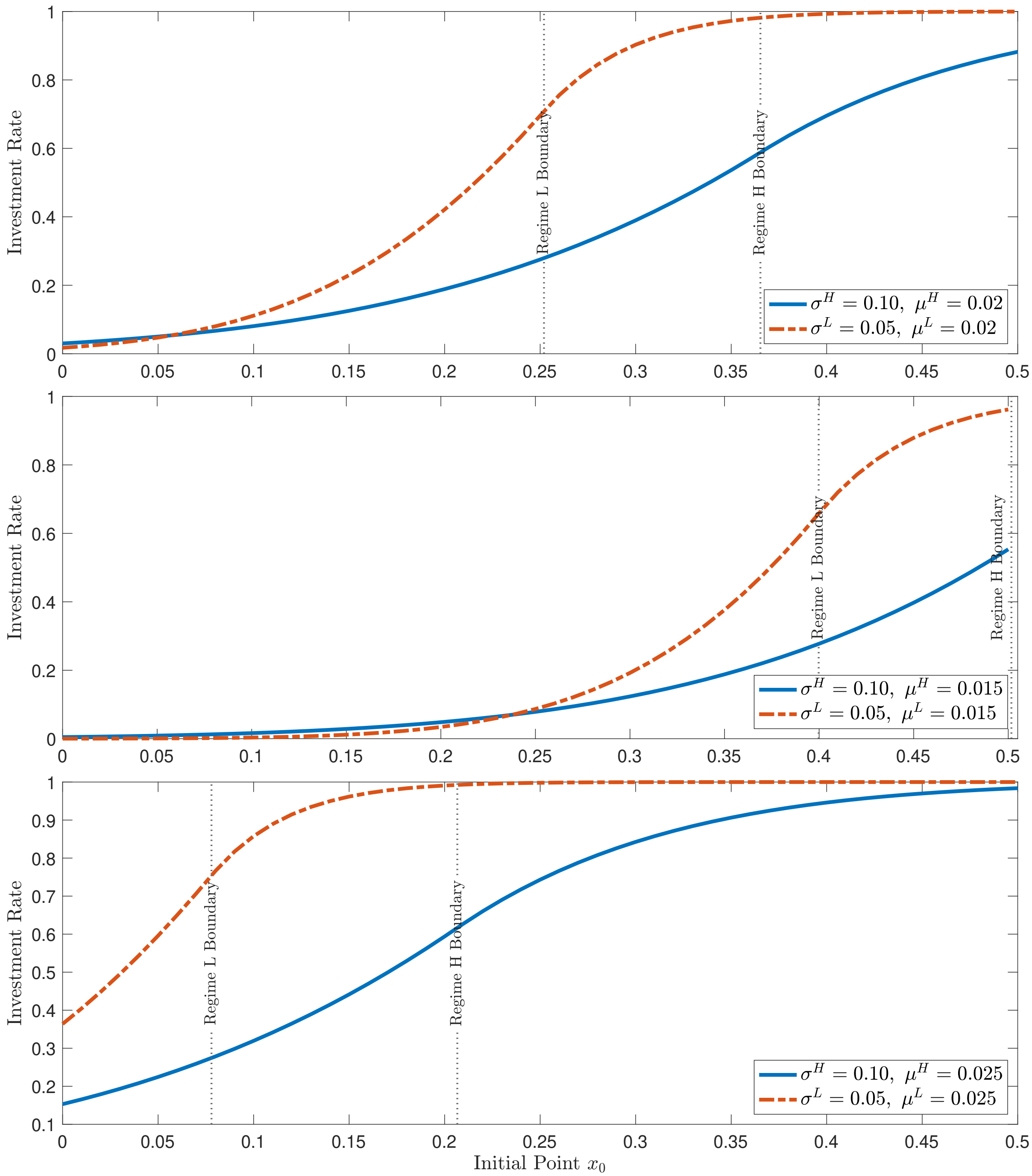


Figure 4: **Investment rate with stochastic volatility as a function of the initial point x_0, fixing $T = 5$.** This figure plots the average investment rate over 5 years for different initial values of x_0. In contrast to the next figure, $\mu^H = \mu^L$ in all cases.

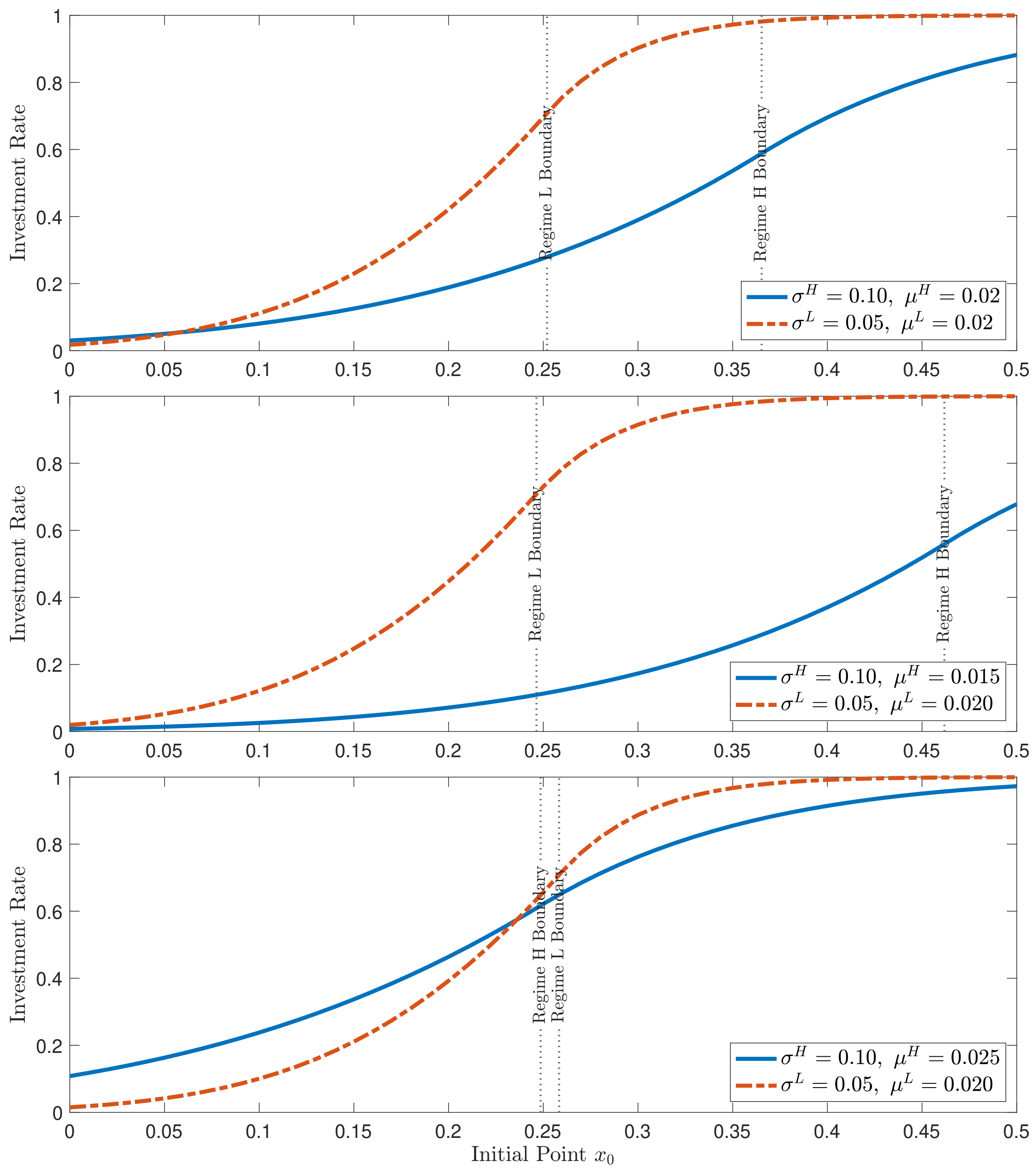
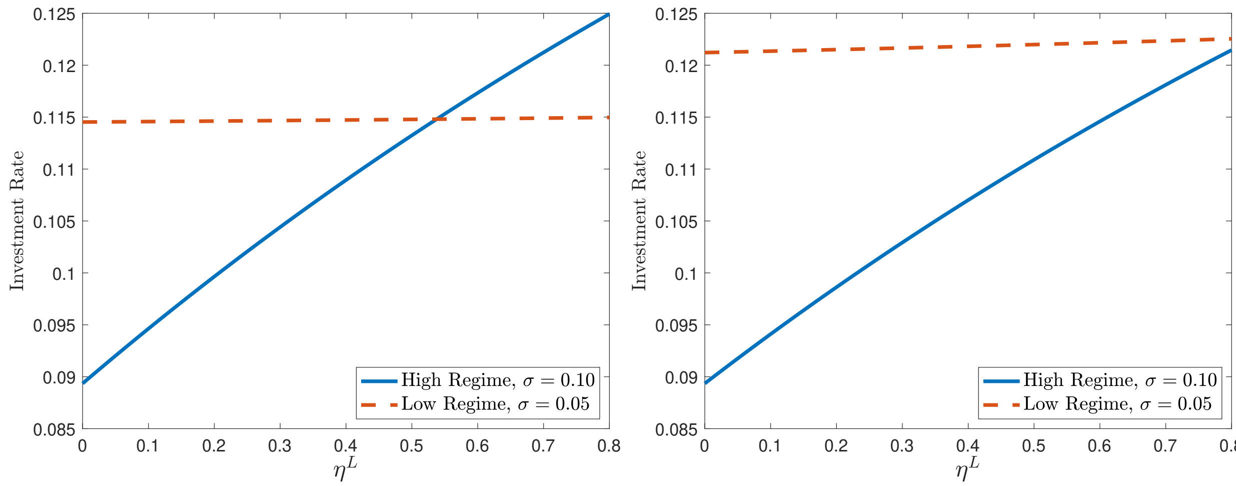


Figure 5: **Investment rate with stochastic volatility as a function of the initial point x_0, fixing $T = 5$.** This figure plots the average investment rate over 5 years for different initial values of x_0. Importantly, it separates investment into the following cases: $\mu^H = \mu^L,\ \mu^H > \mu^L$, and $\mu^H < \mu^L$.



(a) Fixing $\eta^H = 0.05$ (b) Fixing $\eta^H = 0.20$

Figure 6: **Investment rate with stochastic volatility as a function of the arrival rate of the low regime η^L.** This figure shows the investment probability strictly increasing in the arrival rate of the low volatility regime while holding T fixed at 5 years and $\mu^H = \mu^L$.

2.4 The Consumer-Worker-Investor

This section addresses the concern that firms' cost of capital change in response to uncertainty shocks. I first present the setup of the model in Section 2.4.1 and derive results under static volatility, avoiding analysis since the intuition from the previous sections carry over. In Section 2.4.2, I move on to stochastic volatility.

2.4.1 Preferences and Technology

Suppose there exists a representative consumer-worker-investor who maximizes

$$\max_{c_s, \mathbf{d}l_s} \ \mathbb{E}_t \left[\int_t^\infty e^{-\rho(s-t)} \frac{c_s^{1-\gamma} - 1}{1-\gamma} \mathbf{d}s - \int_t^\infty e^{-\rho(s-t)} \vartheta_s \mathbf{d}l_s \right]$$

where the time rate of preference is ρ, risk aversion is γ and consumption is c_t. Importantly, whereas the previous section used q as an fixed investment cost in units of output, ϑ is disutility from capital installation, which requires labor. I assume that labor is not required in the production of the good but only in the installation of new capital when a firm adopts a project.[5]

Furthermore, suppose there is a continuum of firms in unit measure indexed by i. Each firm produces a consumption good following the output process $y_t^i = \zeta_{n,t}^i A_t^i$ where $\zeta_{n,t}^i \doteq [\alpha(z_0^i + \sum_n \mathfrak{z}_n)^\varepsilon + (1-\alpha)(k_0^i + \sum_n \mathfrak{k}_n)^\varepsilon]^{1/\varepsilon}$. To explain notation, (z_0^i, k_0^i) denotes firm i's

[5] This assumption prevents lumpiness in consumption whenever a mass of firms adopt a project.

initial capital stock at date 0, and $(\mathfrak{z}_n, \mathscr{k}_n)$ denotes capital accrued from the nth project completed by time t. Moreover, assume each firm is driven by an independent Brownian motion,

$$\mathbf{d}A^i/A^i = \mu\mathbf{d}t + \sigma^i\left(\sqrt{\varrho}\mathbf{d}Z_t^* + \sqrt{1-\varrho}\mathbf{d}Z_t^i\right)$$

where $\mathbf{d}Z_t^*$ is aggregate risk and $\mathbf{d}Z_t^i$ is firm-specific risk. For simplicity, assume μ and ϱ are common across all firms and $\mathbf{d}Z_t^i\mathbf{d}Z_t^j = 0\ \forall i \neq j$. However, σ^i is drawn independently from a Uniform distribution on $[\sigma_*, \sigma^*]$.

As before, investment projects arrive exogenously at rate $\lambda\mathbf{d}t$, independently across firms. By the law of large numbers, there is a mass of λ firms who receive a project at a given point in time. Moreover, with probability 1, there is an upper-limit of λt projects that are adopted by time t.

Since the representative consumer-worker-investor consumes all output, $c_t = y_t \doteq \int_0^1 y^i\mathbf{d}i$, the below lemma characterizes expected consumption after applying the change of variable, $x = \ln A$.

Lemma 4. Expected consumption at t is

$$\mathbb{E}_t\left[y_t\right] = e^{x_t} \times \frac{1}{\sigma^* - \sigma_*}\int_{\sigma_*}^{\sigma^*}\sum_{n=0}^{\lambda t}\zeta_n\frac{\Lambda(t,\sigma)^n e^{-\Lambda(t,\sigma)}}{n!}\mathbf{d}\sigma \tag{10}$$

where

$$\Lambda(t,\sigma) = \lambda\sum_{n=1}^{\lambda t}\int_0^t\int_{-\infty}^{\infty}\Phi\left(-\frac{x_n^* - (\mu - .5\sigma^2)s}{\sigma\sqrt{s}}\right)f_{x_n^*}(x_n^*)\mathbf{d}x^*\mathbf{d}s$$

and $f_{x_n^*}$ is the density function for the distribution of stopping times x^* for the nth project and $\Phi(\cdot)$ is the normal cumulative distribution function for the distribution of first passage time to x^*. Furthermore, define $w_t^i \doteq y_t^i/y_t$. The stochastic differential equation for $x_t \doteq \int_0^1 x_t^i\mathbf{d}i$ is given by

$$\mathbf{d}x_t = \left(\mu - \frac{1}{2}\varrho\bar{\sigma}_t^2\right)\mathbf{d}t + \bar{\sigma}_t\sqrt{\varrho}\mathbf{d}Z_t^*$$

where $\bar{\sigma}_t = \int_0^1 w_t^i\sigma^i$.

To complete the formulation of firm i's problem, it is important to specify ϑ_t, the disutility from labor required to install new capital. I follow Gârleanu et al. 2012 who consider a similar set-up in which a firm solves an optimal stopping problem to "plant" a new Lucas tree after the arrival of a new technological epoch. In their model, the representative consumer-worker-investor provides competitive labor to plant a tree, and as the owner of all firms, both pays and receives wage payments. The authors specify wages and the disutility from labor to match some stylized facts in the data. Namely, labor income must be cointegrated with total output. Moreover, by the envelope condition in dynamic programming, wage payments q_t must be given by $q_t = \frac{\vartheta_t}{U_c}$ where U_c is marginal utility of consumption. Further requiring that successive trees must be more costly to plant, the authors specify a functional form for ϑ_t. Adapting their approach to this setting, I set $\vartheta_t = \vartheta U_c y_t^{\gamma-(1-\gamma)/\vartheta}$.

Using $V(x^i, \zeta_n^i)$ to denote firm i's value function (the subscript n denotes the number of projects completed by t, $\zeta_{n,t}^i \doteq [\alpha(z_0^i + \sum_n \mathfrak{z}_n)^\varepsilon + (1-\alpha)(k_0^i + \sum_n \mathfrak{k}_n)^\varepsilon]^{1/\varepsilon}$), the HJB equation in the no-action region between stopping times τ_n and τ_{n+1} can be written,

$$\begin{aligned}\rho V(x^i, \zeta_n^i) = y^{-\gamma}\zeta_n^i e^{x^i} + \frac{\partial V(x^i, \zeta_n^i)}{\partial x^i}\left(\mu - \frac{1}{2}(\sigma^i)^2\right) + \frac{1}{2}\frac{\partial^2 V(x^i, \zeta_n^i)}{\partial (x^i)^2}(\sigma^i)^2 \\ + \lambda\Big(V(x^i, \zeta_{n+1}^i) - V(x^i, \zeta_n^i)\Big)\end{aligned}$$

subject to the boundary conditions

$$\begin{aligned}V(x^i, \zeta_n^i) &= V(x^i, \zeta_{n+1}^i) - \vartheta y^{(\gamma-1)/\vartheta} \\ \frac{\partial V(x^i, \zeta_n^i)}{\partial x^i} &= \frac{\partial V(x^i, \zeta_{n+1}^i)}{\partial x^i}\end{aligned}$$

The following corollary describes the investment threshold for the $n+1$th project.

Corollary 2. Let $\delta \doteq r - \mu + \frac{1}{2}(\sigma^i)^2$. The optimal stopping time for adoption of the $n+1$th project is the first time

$$\tau_{n+1}^* \doteq \inf\left\{t : x_t \geq \log\left(\frac{\phi_1^+}{\phi_1^+ - 1}\frac{\vartheta y^{(\gamma-1)/\vartheta - 1}}{\frac{\zeta_{n+1}^i - \zeta_n^i + \frac{\lambda}{\delta}(\tilde{\zeta}_{n+2}^i - \tilde{\zeta}_{n+1}^i)}{\delta+\lambda}}\right) = \widehat{x}\right\} \tag{11}$$

where $\phi_1 \pm = -(\mu - .5(\sigma^i)^2)/(\sigma^i)^2 \pm \sqrt{2(\rho+\lambda)(\sigma^i)^2 + (\mu - .5(\sigma^i)^2)^2}/(\sigma^i)^2$

The equilibrium riskless rate is given by the drift of the consumer-worker-investor's marginal utility over consumption,

$$r_t = \rho + \gamma\left[\left(\mu - \frac{1}{2}(\gamma+1)\varrho\bar{\sigma}_t^2\right) + \Lambda(t, \bar{\sigma}) \times \Delta_t\right] \tag{12}$$

where

$$\Delta_t \doteq \frac{1}{\sigma^* - \sigma_*}\int_{\sigma_*}^{\sigma^*}\sum_{n=0}^{\lambda t}(\zeta_{n+1} - \zeta_n)\frac{\Lambda(t,\sigma)^n e^{-\Lambda(t,\sigma)}}{n!}\mathbf{d}\sigma$$

The riskless rate is the sum of the following: the time rate of preference, a term proportional to the drift of output, a term related to prudence and precautionary demand, and a final term that captures the expected marginal increase in aggregate output over the next time interval due to the mass of firms that complete a new project.

2.4.2 Stochastic Volatility under General Equilibrium

As Section 2.3 and Appendix 5.2 provide a detailed methodology on deriving closed-form solutions under stochastic volatility, I adapt the same setup as before and state the results. Importantly, a regime change applies to all firms and firm-level volatility in the high regime is a flat increase of ε across all firms. First, the HJB equations can be written

$$
\begin{aligned}
\rho V^H(x^i,\zeta^i_n) &= y^{-\gamma}\zeta^i_n e^{x^i} + \frac{\partial V^H(x^i,\zeta^i_n)}{\partial x^i}\left(\mu - \frac{1}{2}(\sigma^{i,H})^2\right) + \frac{1}{2}\frac{\partial^2 V(x^i,\zeta^i_n)}{\partial (x^i)^2}(\sigma^{i,H})^2 \\
&\quad + \lambda\Big(V^H(x^i,\zeta^i_{n+1}) - V^H(x^i,\zeta^i_n)\Big) + \eta^L\Big(V^L(x^i,\zeta^i_n) - V^H(x^i,\zeta^i_n)\Big) \\
\rho V^L(x^i,\zeta^i_n) &= y^{-\gamma}\zeta^i_n e^{x^i} + \frac{\partial V^L(x^i,\zeta^i_n)}{\partial x^i}\left(\mu - \frac{1}{2}(\sigma^{i,L})^2\right) + \frac{1}{2}\frac{\partial^2 V(x^i,\zeta^i_n)}{\partial (x^i)^2}(\sigma^{i,L})^2 \\
&\quad + \lambda\Big(V^L(x^i,\zeta^i_{n+1}) - V^L(x^i,\zeta^i_n)\Big) + \eta^H\Big(V^H(x^i,\zeta^i_n) - V^L(x^i,\zeta^i_n)\Big)
\end{aligned}
$$

subject to the boundary conditions

$$
\begin{aligned}
&V^H(x^i,\zeta^i_n) = V^H(x^i,\zeta^i_{n+1}) - \vartheta y^{(\gamma-1)/\vartheta} \\
&V^L(x^i,\zeta^i_n) = V^L(x^i,\zeta^i_{n+1}) - \vartheta y^{(\gamma-1)/\vartheta} \\
&\frac{\partial V^H(x^i,\zeta^i_n)}{\partial x^i} = \frac{\partial V^H(x^i,\zeta^i_{n+1})}{\partial x^i} \\
&\frac{\partial V^L(x^i,\zeta^i_n)}{\partial x^i} = \frac{\partial V^L(x^i,\zeta^i_{n+1})}{\partial x^i}
\end{aligned}
$$

Similar to before, there are constants $\mathfrak{a}_1, \mathfrak{a}_3, \mathfrak{b}_1, \mathfrak{b}_3$, eigenvalues $\mathfrak{w}^{(1)}, \mathfrak{w}^{(3)}$ and elements of eigenvectors $\mathbf{v}^{(1)}, \mathbf{v}^{(3)}$ corresponding to these eigenvalues such that,

Proposition 4. In volatility regime H, investment occurs the first time

$$
\tau^{i,H*} \doteq \inf\left\{ t : x^i_t \geq \log\left(\frac{\mathfrak{w}^{(1)}}{\mathfrak{w}^{(1)} - 1} \frac{\mathfrak{v}^{(3)}_1 - \mathfrak{v}^{(3)}_3}{\mathfrak{v}^{(3)}_1(\mathfrak{b}_3 - \mathfrak{a}_3) - \mathfrak{v}^{(3)}_3(\mathfrak{b}_1 - \mathfrak{a}_1)} \times y^{(\gamma-1)/\vartheta} \right) = \widehat{x}^{i,H} \right\} \tag{13}
$$

In volatility regime L, investment occurs the first time

$$
\tau^{i,L*} \doteq \inf\left\{ t : x^i_t \geq \log\left(\frac{\mathfrak{w}^{(3)}}{\mathfrak{w}^{(3)} - 1} \frac{\mathfrak{v}^{(1)}_1 - \mathfrak{v}^{(1)}_3}{\mathfrak{v}^{(1)}_1(\mathfrak{b}_3 - \mathfrak{a}_3) - \mathfrak{v}^{(1)}_3(\mathfrak{b}_1 - \mathfrak{a}_1)} \times y^{(\gamma-1)/\vartheta} \right) = \widehat{x}^{i,L} \right\} \tag{14}
$$

The equilibrium riskless rate is again given by

$$r_t = \rho + \gamma \left[\left(\mu - \frac{1}{2}(\gamma+1)\varrho\bar{\sigma}_t^2 \right) + \Lambda(t,\bar{\sigma}) \times \Delta_t^{\{L,H\}} \right] \tag{15}$$

but $\Lambda(t,\bar{\sigma}) \times \Delta_t^{\{L,H\}}$, which expresses the expected marginal increase in output over the next time interval given the current regime must be completed via simulation, which requires an estimate of the number of projects that were completed between $(0,t)$ across several regime changes.

Figure 7 plots average investment rates across firms following a regime change, pooled over $10,000$ sample paths across 500 firms. Initial parameter values are chosen such that unconditionally, the average investment rate is around 0.22 when $r = 0.025$ and risk aversion $\gamma = 2$ to match the data. Because variation in the equilibrium riskless rate is mostly driven by variation in the expected marginal change to production via investment, varying the initial level of r is a good proxy to capture the initial state of the economy. For example, low values of r may indicate that the *aggregate* distance to investment is relatively far, growth is stagnant, and corporate discount rates are subsequently low to entice firms to invest. I first set the initial value of r and track subsequent investment over the next 5 years while updating r_t according to Equation (15).

In the above figure, the yellow dots track average investment following the arrival of a high volatility regime, while the dark blue does so following the arrival of a low regime. What is surprising is that despite fixing μ constant across regime changes, the figure shows that a jump in volatility increases subsequent investment when r is low. Recall that this did not appear to be possible in the partial-equilibrium framework in Section 2.3 unless $\mu^H > \mu^L$. The most plausible explanation for this difference is that when the initial value of r is low, the investment boundary does not shift outward substantially in response to an uncertainty shock, allowing the firm to benefit from the larger price movements induced by high volatility. In addition, the equilibrium riskless rate r_t−which influences the position of the boundary−adjusts only gradually (since the initial aggregate distance was far), giving the firm ample time to capitalize on this opportunity. On the other hand, when the initial level of r is high (which can be interpreted as an overheated economy), an uncertainty shock leads to a sharp drop in subsequent investment.

To summarize, this section highlights the importance of determining the firm's cost of capital in equilibrium. Not only does its cost of capital change in response to a temporary increase in uncertainty, it also changes in response to the initial state of the economy. Both factors directly influence the placement of its optimal investment threshold, changing the impact of uncertainty shocks. In particular, the general equilibrium framework predicts that an uncertainty shock in a sluggish economy can increase aggregate investment over the long

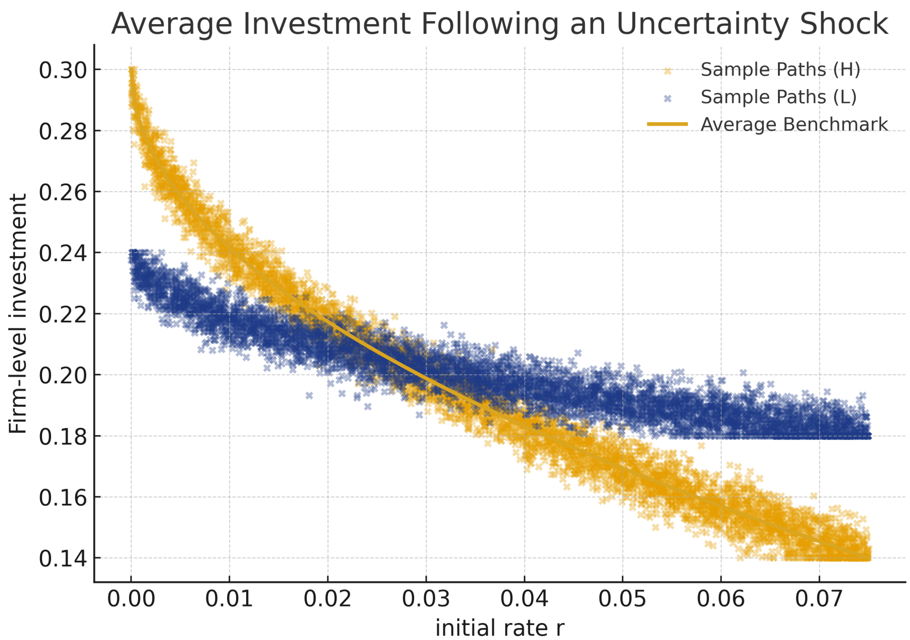


Figure 7: **Investment rate with stochastic volatility as a function of the initial riskless rate r, fixing $T = 5$.** This figure plots the average investment rate over 5 years for different initial values of r in response to a regime change, subsequently updating r_t according to Equation (15). The yellow dots indicates sample paths following the arrival of a high volatility regime while the blue follows investment following the arrival of a low volatility regime.

run, while decrease it in a growing economy.

3 Empirical Analysis

To construct a measure of the distance to investment, $|x_0 - \widehat{x}|$, I estimate a reduced-form model to identify the likelihood that the firm invests given its characteristics, then measure distance as 1 minus this likelihood. More formally, in the first step, I estimate the probit model:

$$\Pr(\mathbb{1}_{\mathrm{ik}_{t+1}} | \mathbf{x}_{i,t}) = \Phi(\mathbf{x}_{i,t}\xi + \delta_i + \gamma_{j,t} + \varepsilon_{i,t+1})$$

where the dependent variable $\mathbb{1}_{\mathrm{ik}_{t+1}}$ takes the value of 1 if the firm's investment rate at $t+1$ is in the top quintile within that industry-year. The independent variables include a vector of firm characteristics $\mathbf{x}_{i,t}$, firm-effects and industry$\times$year fixed effects.[6]

In practice, less than 1% of all firm-year observations in Compustat between 1975-2023

[6] To address issues with convergence and because some industry-years are sparsely populated, I focus estimation on the set of manufacturing firms (2-digit SIC code between 20 and 39), then extend the estimated coefficients to the full sample.

Table 1: This table is the panel regression:

$$\text{Investment rate}_{i,t+1} = \mathbf{x}_{i,t}\boldsymbol{\xi} + \beta_1 \text{Volatility}_{i,t} + \beta_2 \text{InvDist}_t + \beta_3 \text{Volatility}_{i,t} \times \text{InvDist}_t + \delta_i + \gamma_{j,t} + \varepsilon_{i,t+1}$$

Dependent Variable: Investment rate$_{i,t+1}$

	(1)	(2)	(3)
Vol_{1y}	-2.06***	-2.06***	-4.82***
	(-6.14)	(-6.14)	(-3.69)
InvDist		-7.13***	-7.54***
		(-4.26)	(-4.48)
$\text{Vol}_{1y} \times \text{InvDist}$			1.79**
			(2.40)
Firm controls		Yes	
Fixed effects		Firm, sic-2 × Year	
Standard errors		sic-2, Year	
Adjusted R^2	0.54	0.54	0.54
Observations	73,170	73,170	73,170

t statistics in parentheses
* $p < 0.10$
** $p < 0.05$
*** $p < 0.01$

have an investment rate of 0%. As a result, to take the model to the data, we must assume that the firm is in the inaction region for some set of projects but not for others. Pushing this further, we can reasonably assume that firms who have higher investment rates are likely to be in the action region for a greater set of their projects. Hence, the dependent variable focuses on firms who invests a lot.

I then treat the coefficients ξ from this maximum likelihood estimation as constant, structural parameters and construct a new variable:

$$\widehat{\Pr(\mathbb{1}_{\text{ik}_{t+1}})} = \mathbf{x}_{i,t}\xi \tag{16}$$

Finally, the distance to investment is

$$\text{InvDist}_t = 1 - \widehat{\Pr(\mathbb{1}_{\text{ik}_{t+1}})}$$

Hence, the variable looks at firm characteristics at t, and using the parameters estimated in

Table 2: This table is the panel regression:

$$\text{Investment rate}_{i,t+1} = \mathbf{x}_{i,t}\boldsymbol{\xi} + \beta_1 \text{Volatility}_{i,t} + \beta_2 \text{InvDist}_t + \beta_3 \text{Volatility}_{i,t} \times \text{InvDist}_t + \delta_i + \gamma_{j,t} + \varepsilon_{i,t+1}$$

In this regression, the variable InvDist_t is estimated out-of-sample, using the first 10 years as the initial data.

Dependent Variable: Investment rate$_{i,t+1}$			
	(1)	(2)	(3)
ΔVol_{1y}	-3.24***	-3.24***	-7.42**
	(-3.93)	(-3.92)	(-2.56)
InvDist		-6.22**	-6.22**
		(-3.15)	(-2.83)
$\Delta\text{Vol}_{1y} \times$ InvDist			3.07**
			(2.21)
Firm controls		Yes	
Fixed effects		Firm, sic-2	
Standard errors		sic-2, Year	
Adjusted R^2	0.16	0.16	0.17
Observations	56,331	56,331	56,331

t statistics in parentheses
* $p < 0.10$
** $p < 0.05$
*** $p < 0.01$

(16), creates a proxy for the distance $x_t - \widehat{x}$. I then run the following regression:

$$\begin{aligned} \text{Investment rate}_{i,t+1} = \mathbf{x}_{i,t}\boldsymbol{\xi} + \beta_1 \text{Volatility}_{i,t} + \beta_2 \text{InvDist}_t \\ + \beta_3 \text{Volatility}_{i,t} \times \text{InvDist}_t + \delta_i + \gamma_{j,t} + \varepsilon_{i,t+1} \end{aligned} \tag{17}$$

The coefficient of interest is β_3 in the above regression; it is the empirical test of Figure 2: high volatility is associated with higher average investment, but only if the firm is far away from the investment threshold. Table 1 shows the results. As predicted by the theory, the coefficient β_3 is negative and is both economically and statistically significant. The negative coefficient for β_2 makes sense as well. For example firms whose projects that are mostly deep out-of-money will likely invest less, all else equal.

One concern is that the variable InvDist should be measured out-of-sample to avoid a potential bias. In Table 2, I re-run the regression using rolling windows, using the first 10 years of data and accumulating information progressively. The sign of the coefficients are the same though the magnitudes slightly differ, indicating that these results are not mechanically driven.

4 Conclusion

This paper presents a unifying framework to reconcile the conflicting facts in the investment and uncertainty literature, both in the time-series and in the cross-section. I derive closed-form solutions for the optimal investment rate and show using comparative statics how a simple statistic can be used to provide guidelines on when one can expect to see a positive or negative relationship. Turning to general equilibrium under stochastic volatility, I show that endogenizing the firm's cost of capital drastically changes predictions on the impact of uncertainty shocks on investment.

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5 Appendix: Proofs

5.1 Proof of Lemma 2

Proof. By Dynkin's formula (see Fleming and Soner 2006),

$$\mathbb{E}_t\left[f(T,x)\right] = f(t,x) + \mathbb{E}_t\left[\int_t^T \left(\mathscr{D}(f(s,x)) + \frac{\partial f(s,x)}{\partial s}\right]\right)\mathbf{d}s\right]$$

Using the boundary condition $f(T,x)=0$ and the definition of $f(t,x)$,

$$\mathbb{E}_t\left[\int_t^T \left(\mathscr{D}(f^H(s,x)) + \frac{\partial f^H(s,x)}{\partial s}\right)\mathbf{d}s\right] = -\mathbb{E}_t\left[\int_t^T \mathbb{1}_{x_s\geq\widehat{x}}\mathbf{d}s\right]$$

Bringing the expectations operator inside the integral via Fubini's theorem, applying the Radon-Nikodym theorem and by Lebesgue differentiation, we have

$$\mathscr{D}(f^H(t,x)) + \frac{\partial f^H(t,x)}{\partial t} = -\mathbb{1}_{x>\widehat{x}} \quad \text{a.e.}$$

□

5.2 Proof of Proposition 1

Starting from

$$-\xi\overline{f}(\xi,x) + \left[\mu - \frac{1}{2}\sigma^2\right]\frac{\partial\overline{f}(\xi,x)}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2\overline{f}(\xi,x)}{\partial x^2} = -\frac{1}{\xi}\mathbb{1}_{x>\widehat{x}}$$

The ODE splits into two regions: $x<\widehat{x}$ and $x>\widehat{x}$. The general solution is given by

$$\overline{f}(\xi,x) = \begin{cases} A_1 e^{\lambda^+(\xi)x} + A_2 e^{\lambda^-(\xi)x} & \text{if } x<\widehat{x} \\ \frac{1}{\xi^2} + B_1 e^{\lambda^+(\xi)x} + B_2 e^{\lambda^-(\xi)x} & \text{if } x>\widehat{x} \end{cases}$$

where $\lambda^{\pm}(\xi) = -(\mu-0.5\sigma^2)/\sigma^2 \pm 2\sqrt{(\mu-0.5\sigma^2)+2\sigma^2\xi}/\sigma^2$ and $\lambda^{\pm}(\xi)$ emphasizes the fact that the solution to the characteristic equation is a function of the Laplace variable ξ. Economically, there are two boundary conditions,

$$\overline{f}(\xi,-\infty) = 0$$
$$\overline{f}(\xi,\infty) = 1/\xi^2$$

The first condition says that as x moves infinitely below $\widehat{x}$, the expected occupation time above $\widehat{x}$ is equal to zero. The second condition comes from the boundary condition $f(T-t,\infty) = T-t$ by a similar argument and applying the Laplace transform, $\int_0^\infty \tau e^{-\xi\tau}\mathbf{d}\tau = 1/\xi^2$. The two conditions imply

$$\overline{f}(\xi,x) = \begin{cases} A_1 e^{\lambda^+(\xi)x} & \text{if } x < \widehat{x} \\ \frac{1}{\xi^2} + B_2 e^{\lambda^-(\xi)x} & \text{if } x > \widehat{x} \end{cases}$$

and matching the value and derivative at $\widehat{x}$,

$$\overline{f}(\xi,x) = \begin{cases} -\frac{1}{2\xi^2}\left(1 + \frac{1}{\sqrt{1+2\left(\frac{\sigma}{\mu}\right)^2\xi}}\right) e^{\lambda^+(\xi)(x-\widehat{x})} & \text{if } x < \widehat{x} \\ \frac{1}{\xi^2} - \frac{1}{2\xi^2}\left(1 + \frac{1}{\sqrt{1+2\left(\frac{\sigma}{\mu}\right)^2\xi}}\right) e^{\lambda^-(\xi)(x-\widehat{x})} & \text{if } x > \widehat{x} \end{cases} \tag{18}$$

With explicit solutions for $\overline{f}(\xi,x)$, all that remains is to invert them. In general, the inverse Laplace transform is given by the Bromwich integral,

$$f(\tau,x) = \mathscr{L}^{-1}\{\overline{f}(\xi,x)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\xi\tau}\overline{f}(\xi,x)\mathbf{d}\xi \tag{19}$$

To proceed further, it is necessary to first state some results from complex analysis.

Remark 1. Let $f: D \to \mathbb{C}$ be an analytic function which has a pole at ξ_0, and $D \subset \mathbb{C}$. Then,

$$\lim_{\xi\to\xi_0} |f(\xi)| = \infty.$$

For our purposes, we can consider the definition of a pole as a point at which the function is not well-behaved, and singularities in general as such points. Moreover,

Definition 1 (Residue)**.** Let $f: \dot{U}_r(\xi_0) \to \mathbb{C}$ be an analytic function, such that ξ_0 is a singularity of f, and let

$$f(\xi) = \sum_{n=-\infty}^{\infty} a_n(\xi-\xi_0)^n$$

be its Laurent series in $\dot{U}_r(\xi_0)$ where $\dot{U}_r(\xi_0) \doteq \{\xi \in \mathbb{C} : 0 < |\xi-\xi_0| < r\}$ is a punctured disk centered at ξ_0. The coefficient a_{-1} in this expansion is called the residue of f at the point ξ_0.

For well-behaved functions that do not possess singularities, the coefficients $a_{-1}, a_{-2}, ...$ simply equal zero and the above representation reduces to its Taylor series. Finally,

Theorem 5 (The Residue Theorem)**.** Let $f(\xi)$ be analytic inside and on a simple closed contour Ω, except for a finite number of isolated singular points $\xi_1, \xi_2, ..., \xi_N$ located inside Ω. Then,

$$\oint_\Omega f(\xi)\mathbf{d}\xi = 2\pi i \sum_j^N \mathrm{Res}\{f(\xi); \xi_j\}$$

Together, the above definitions and results tell us that if we can successfully close the path of integration for the Bromwich contour in Equation (19), we can further analyze its properties by simply evaluating the behavior of the integrand at its singular points.[7] The next proposition applies the residue theorem (among other things) to our setting.

Proof. By inspection of Equation (18), $\bar{f}(\xi, x)$ has a (second-order) pole at $\xi_0 = 0$. As a result, we can choose an arbitrary value of $\gamma > 0$ and close the contour along the left-axis of the complex plane while enclosing the point ξ_0. Figure IA.8 illustrates the path of integration.

Importantly, we need to ensure the analyticity of $\bar{f}(\xi, x)$ in and on the closed contour Ω. Because $\bar{f}(\xi, x)$ involves the square root of a complex number ξ in the characteristic equation for $\lambda(\xi)$, it is in fact multi-valued for values of $\xi \neq -\bar{\mu}/(2\sigma^2)$ as the function approaches the real axis. This will hold more broadly whenever the original function $f(\cdot)$ solves a linear second-order parabolic equation as is often the case in Finance. To address this issue, given the placement of the Bromwich line $\boldsymbol{\gamma}$, the location of the pole ξ_0 and the branch point ξ_b, we can simply cut the plane along the negative real axis and loop around ξ_b as shown in Figure IA.8.

By the Residue theorem,

$$\oint_\Omega e^{\xi\tau}\bar{f}(\xi, x)\mathbf{d}\xi = 2\pi i \times \mathrm{Res}\{e^{\xi\tau}\bar{f}(\xi, x); \xi_0\}$$

but also,

$$\oint_\Omega = \int_{\mathcal{C}_1} + \int_{\mathcal{C}_2} + \int_{\mathcal{C}_3} + \int_{\mathcal{C}_4} + \int_{\delta} + \int_{\gamma}$$

[7]Evaluation of the Bromwich integral via the Residue theorem does not require the function $\bar{f}(\xi, x)$ to possess poles. If the function is analytic everywhere or can be analytically extended by continuation onto $D \cup \{\xi_0\}$, Cauchy's theorem applies and the sum of residues equal zero. In this case, branch points, which will generally exist in our setting, will suffice to evaluate the Bromwich integral meaningfully.

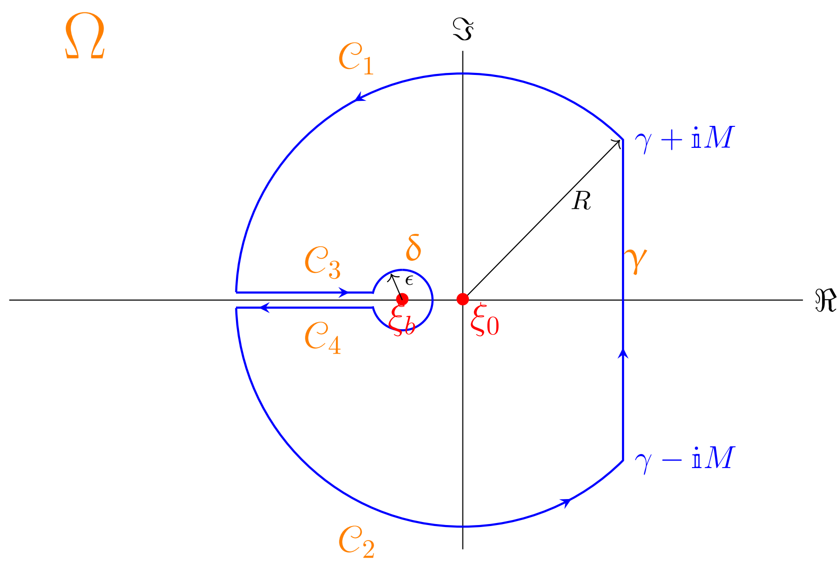


Figure IA.8: **Path of integration with a branch cut around ξ_b and a pole at ξ_0.** This figure plots the closed contour Ω which connects 6 separate contours: $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \delta, \gamma$. The points ξ_0, ξ_b indicate the second-order pole and branch point of the function $\bar{f}(\xi, x)$, and ϵ, R indicate the radius of the semi-circular arcs enclosing these points.

Clearly, $\int_\gamma$ is our object of interest. To match the Bromwich integral in Equation (19), we simply take limits as $R \to \infty$, such that our vertical line γ in Figure IA.8 extends from $\gamma - \mathbb{i}\infty$ to $\gamma + \mathbb{i}\infty$.

In the other direction, we can take $\epsilon \to 0$ such that $\mathcal{C}_3$ and $\mathcal{C}_4$ approach the real-axis from above and below.

Then, by Jordan's lemma,

$$\begin{aligned}\oint_\Omega &= \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \int_{\mathcal{C}_1} + \int_{\mathcal{C}_2} + \int_{\mathcal{C}_3} + \int_{\mathcal{C}_4} + \int_\delta + \int_\gamma \\ &= \int_{\mathcal{C}_3} + \int_{\mathcal{C}_4} + \int_\gamma = \lim_{\epsilon\to 0^+} \int_{-\infty}^{\xi_b} + \lim_{\epsilon\to 0^-} \int_{\xi_b}^{-\infty} + \int_{\gamma-\mathbb{i}\infty}^{\gamma+\mathbb{i}\infty}\end{aligned}$$

which implies that

$$\int_{\gamma-\mathbb{i}\infty}^{\gamma+\mathbb{i}\infty} = \oint_\Omega - \lim_{\epsilon\to 0^+} \int_{\mathcal{C}_3} - \lim_{\epsilon\to 0^-} \int_{\mathcal{C}_4}$$

or,

$$\frac{1}{2\pi \mathbb{i}} \int_{\gamma-\mathbb{i}\infty}^{\gamma+\mathbb{i}\infty} e^{\xi\tau}\overline{f}(\xi,x)\mathbf{d}\xi = \mathrm{Res}\{e^{\xi\tau}\overline{f}(\xi,x);\xi_0\}$$
$$-\frac{1}{2\pi \mathbb{i}}\left(\lim_{\epsilon\to 0^+}\int_{-\infty}^{\xi_b} e^{\xi\tau}\overline{f}(\xi,x)\mathbf{d}\xi + \lim_{\epsilon\to 0^-}\int_{\xi_b}^{-\infty} e^{\xi\tau}\overline{f}(\xi,x)\mathbf{d}\xi\right)$$

To proceed, I first compute the residue at $\xi_0 = 0$, then calculate the branch cut contribution separately. The term $e^{\xi\tau}\overline{f}(\xi,x)$ can be written,

$$e^{\xi\tau}\overline{f}(\xi,x) = \frac{1}{2\xi^2}\left(1+\frac{1}{\sqrt{1+2\left(\frac{\sigma}{\bar{\mu}}\right)^2\xi}}\right)\times\exp\left(\xi\tau+\left(\frac{-\bar{\mu}}{\sigma^2}+\frac{2\sqrt{\bar{\mu}+2\sigma^2\xi}}{\sigma^2}\right)(x-\widehat{x})\right)$$

To compute the residue at ξ_0, we express $e^{\xi\tau}\overline{f}(\xi,x)$ in its Laurent expansion around ξ_0 and find the coefficient of $\frac{1}{\xi-\xi_0}$, corresponding to the a_{-1} term in the definition of a residue. Expanding term by term,

$$\exp\left[\left(\frac{-\bar{\mu}}{\sigma^2}+\frac{2\sqrt{\bar{\mu}+2\sigma^2\xi}}{\sigma^2}\right)(x-\widehat{x})\right] \approx 1-\frac{2(\widehat{x}-x)}{\bar{\mu}}\xi+\frac{2(\widehat{x}-x)\left(\sigma^2\bar{\mu}+2\bar{\mu}^2(\widehat{x}-x)\right)}{\bar{\mu}^4}\xi^2+\cdots$$

$$\frac{1}{2}\left(1+\frac{1}{\sqrt{1+2\left(\frac{\sigma}{\bar{\mu}}\right)^2\xi}}\right) \approx 1-\frac{1}{2}\left(\frac{\sigma}{\bar{\mu}}\right)^2\xi+\frac{3}{4}\left(\frac{\sigma}{\bar{\mu}}\right)^4\xi^2-\frac{5}{4}\left(\frac{\sigma}{\bar{\mu}}\right)^6\xi^3+\cdots$$

$$\exp(\xi\tau) \approx 1+\tau\xi+\frac{1}{2}\tau^2\xi^2+\frac{1}{6}\tau^3\xi^3+\cdots$$

Collecting and simplifying, we have

$$\mathrm{Res}\{e^{\xi\tau}\overline{f}(\xi,x);\xi_0\} = \tau-\frac{2(\widehat{x}-x)}{\bar{\mu}}-\frac{1}{2}\frac{\sigma^2}{\bar{\mu}^2}$$

Next, I consider the integral: $\lim_{\epsilon\to 0^+}\frac{1}{2\pi\mathbb{i}}\int_{-\infty}^{\xi_b} e^{\xi\tau}\overline{f}(\xi,x)\mathbf{d}\xi$,

$$= \lim_{\epsilon\to 0^+}\frac{1}{2\pi\mathbb{i}}\int_{-\infty}^{\xi_b}\frac{1}{2\xi^2}\left(1+\frac{1}{\sqrt{1+2\left(\frac{\sigma}{\bar{\mu}}\right)^2\xi}}\right)\times\exp\left(\xi\tau+\left(\frac{-\bar{\mu}}{\sigma^2}+\frac{2\sqrt{\bar{\mu}+2\sigma^2\xi}}{\sigma^2}\right)(x-\widehat{x})\right)\mathbf{d}\xi$$

Since $\xi_b = -\bar{\mu}/(2\sigma^2)$ and the branch cut runs along $(-\infty, \xi_b)$, we must express ξ in polar coordinates inside square roots. Taking $\arg(z) \in (-\pi, \pi)$,

$$
\begin{aligned}
&= \frac{1}{2\pi\mathbb{i}} \int_{-\infty}^{\xi_b} \frac{1}{2\xi^2} \left(1 + \frac{1}{\sqrt{1 + 2\left(\frac{\sigma}{\bar{\mu}}\right)^2 |\xi| e^{\mathbb{i}\pi}}} \right) \times \exp\left(\xi\tau + \left(\frac{-\bar{\mu}}{\sigma^2} + \frac{2\sqrt{\bar{\mu} + 2\sigma^2 |\xi| e^{\mathbb{i}\pi}}}{\sigma^2} \right) (x - \widehat{x}) \right) \mathbf{d}\xi \\
&= \frac{1}{2\pi\mathbb{i}} \int_{-\infty}^{0} \frac{1}{2(\xi + \xi_b)^2} \left(1 - \frac{\mathbb{i}}{\frac{\sigma}{\bar{\mu}}\sqrt{2|\xi|}} \right) \times \exp\left((\xi + \xi_b)\tau + \left(\frac{-\bar{\mu}}{\sigma^2} + \frac{2\mathbb{i}\sqrt{2|\xi|}}{\sigma} \right) (x - \widehat{x}) \right) \mathbf{d}\xi \\
&= \frac{1}{2\pi\mathbb{i}} \int_{\infty}^{0} \frac{1}{2(\xi_b - \xi)^2} \left(1 - \frac{\mathbb{i}}{\frac{\sigma}{\bar{\mu}}\sqrt{2\xi}} \right) \times \exp\left((\xi_b - \xi)\tau + \left(\frac{-\bar{\mu}}{\sigma^2} + \frac{2\mathbb{i}\sqrt{2\xi}}{\sigma} \right) (x - \widehat{x}) \right) \mathbf{d}\xi \\
&= -\frac{1}{2\pi\mathbb{i}} \int_{0}^{\infty} \frac{1}{2(\xi_b - \xi)^2} \left(1 - \frac{\mathbb{i}}{\frac{\sigma}{\bar{\mu}}\sqrt{2\xi}} \right) \times \exp\left((\xi_b - \xi)\tau + \left(\frac{-\bar{\mu}}{\sigma^2} + \frac{2\mathbb{i}\sqrt{2\xi}}{\sigma} \right) (x - \widehat{x}) \right) \mathbf{d}\xi
\end{aligned}
$$

Doing the same for the integral: $\lim_{\epsilon\to 0^-} \frac{1}{2\pi\mathbb{i}} \int_{\xi_b}^{-\infty} e^{\xi\tau} \overline{f}(\xi, x) \mathbf{d}\xi$,

$$
= \frac{1}{2\pi\mathbb{i}} \int_{0}^{\infty} \frac{1}{2(\xi_b - \xi)^2} \left(1 + \frac{\mathbb{i}}{\frac{\sigma}{\bar{\mu}}\sqrt{2\xi}} \right) \times \exp\left((\xi_b - \xi)\tau + \left(\frac{-\bar{\mu}}{\sigma^2} - \frac{2\mathbb{i}\sqrt{2\xi}}{\sigma} \right) (x - \widehat{x}) \right) \mathbf{d}\xi
$$

Taking the difference and using $(1 - a)\exp(\mathbb{i}b) - (1 + a)\exp(-\mathbb{i}b) = 2\mathbb{i}\sin(b) - 2a\cos(b)$,

$$
\begin{aligned}
&- \frac{1}{2\pi\mathbb{i}} \left(\lim_{\epsilon\to 0^+} \int_{-\infty}^{\xi_b} e^{\xi\tau} \overline{f}(\xi, x) \mathbf{d}\xi + \lim_{\epsilon\to 0^-} \int_{\xi_b}^{-\infty} e^{\xi\tau} \overline{f}(\xi, x) \mathbf{d}\xi \right) \\
&= \frac{1}{2\pi} \int_{0}^{\infty} \frac{1}{\xi^2} e^{(\frac{1}{2}\frac{\bar{\mu}^2}{\sigma^2} - \xi)\tau - \frac{2\bar{\mu}}{\sigma^2}(x - \widehat{x})} \times \left[\sin\left(\frac{2\sqrt{2\xi}}{\sigma}(x - \widehat{x}) \right) - \frac{\bar{\mu}}{\sigma\sqrt{2\xi}} \cos\left(\frac{2\sqrt{2\xi}}{\sigma}(x - \widehat{x}) \right) \right] \mathbf{d}\xi
\end{aligned}
$$

Adding the residue and the branch cut contribution concludes the proof. □

5.3 Proof of Proposition 2

The general solution to system of constant coefficient first order linear equations in Equation (7) is

$$
\mathbf{V} = \sum_{i}^{4} c_i \mathbf{v}^{(i)} e^{\omega^{(i)} x}
$$

where c_i are integration constants and $\mathbf{v}^{(i)}$ is the eigenvector associated with the ith eigenvalue $\omega^{(i)}$.

In principle, the eigenvalues can be computed explicitly from $\det(M - \omega I) = 0$ using

Ferrari's method. Though the main body of the text uses exact solutions, for analytic convenience, we can treat M in a block structure which leads to an approximate error of $(\frac{\eta^H}{.5(\sigma^L)^2})(\frac{\eta^L}{.5(\sigma^H)^2})$. Then, the eigenvalues satisfy

$$\left(\omega^2 + \frac{\mu - \kappa\sigma^H - .5(\sigma^H)^2}{.5(\sigma^H)^2}\omega - \frac{(r+1+\eta^L)}{.5(\sigma^H)^2}\right)\left(\omega^2 + \frac{\mu - \kappa\sigma^L - .5(\sigma^L)^2}{.5(\sigma^L)^2}\omega - \frac{(r+1+\eta^H)}{.5(\sigma^L)^2}\right) = 0$$

We have 4 solutions for ω. However, we require $e^{\omega x} \to 0$ as $x \to -\infty$ so we are left with two positive solutions,

$$\omega^{(1)} = -\frac{\mu - \kappa\sigma^H - .5(\sigma^H)^2}{(\sigma^H)^2} + \sqrt{\left(\frac{\mu - \kappa\sigma^H - .5(\sigma^H)^2}{.5(\sigma^H)^2}\right)^2 + 4\left(\frac{r+1+\eta^L}{.5(\sigma^H)^2}\right)}\Big/2$$

and

$$\omega^{(3)} = -\frac{\mu - \kappa\sigma^L - .5(\sigma^L)^2}{(\sigma^L)^2} + \sqrt{\left(\frac{\mu - \kappa\sigma^L - .5(\sigma^L)^2}{.5(\sigma^L)^2}\right)^2 + 4\left(\frac{r+1+\eta^H}{.5(\sigma^L)^2}\right)}\Big/2$$

The eigenvector associated with the ith eigenvalue is

$$\mathbf{v}^{(i)} = \begin{bmatrix} v_1^{(i)} \\ v_2^{(i)} \\ v_3^{(i)} \\ v_4^{(i)} \end{bmatrix} = \begin{bmatrix} v_1^{(i)} \\ \omega^{(i)} v_1^{(i)} \\ v_1^{(i)} \frac{r+1+\eta^L - \left(\mu - \kappa\sigma^H - .5(\sigma^H)^2 + \omega^{(i)}/(.5(\sigma^H)^2)\right)\omega^{(i)}}{\eta^L} \\ \omega^{(i)} v_3^{(i)} \end{bmatrix}$$

Collecting, the value functions V^H, V^L are given by

$$\begin{aligned} V^H &= c_1 v_1^{(1)} e^{\omega^{(1)}x} + c_3 v_1^{(3)} e^{\omega^{(3)}x} + a_1 e^x - \frac{1}{r+1}q \\ V^L &= c_1 v_3^{(1)} e^{\omega^{(1)}x} + c_3 v_3^{(3)} e^{\omega^{(3)}x} + a_3 e^x - \frac{1}{r+1}q \end{aligned}$$

where

$$\begin{aligned} a_1 &= \frac{\eta^L\left(\zeta_0 + \frac{\tilde{\zeta}_1}{\delta^L}\right) + \Gamma^L\left(\zeta_0 + \frac{\tilde{\zeta}_1}{\delta^H}\right)}{\Gamma^L\Gamma^H - \eta^H\eta^L} \\ a_3 &= \frac{\eta^H\left(\zeta_0 + \frac{\tilde{\zeta}_1}{\delta^H}\right) + \Gamma^H\left(\zeta_0 + \frac{\tilde{\zeta}_1}{\delta^L}\right)}{\Gamma^L\Gamma^H - \eta^H\eta^L} \end{aligned}$$

and I define $\Gamma^H \doteq 1 + r + \eta^L - \mu + \kappa\sigma^H$ and $\Gamma^L \doteq 1 + r + \eta^H - \mu + \kappa\sigma^L$. Moreover, let $\zeta_3 \doteq [\alpha(z+\mathfrak{z}+\tilde{\mathfrak{z}})^\varepsilon+(1-\alpha)(k+\Bbbk+\tilde{\Bbbk})^\varepsilon]^{1/\varepsilon}$. Applying the boundary conditions, the integration constants c_1 and c_3 are pinned down by

$$c_1 = \frac{e^{-\omega^{(1)}\widehat{x}}\left[v_1^{(3)}\left((b_3-a_3)e^{\widehat{x}}-q\right)-v_3^{(3)}\left((b_1-a_1)e^{\widehat{x}}-q\right)\right]}{v_1^{(3)}v_3^{(1)}-v_1^{(1)}v_3^{(3)}}$$
$$c_3 = \frac{e^{-\omega^{(3)}\check{x}}\left[v_3^{(1)}\left((b_1-a_1)e^{\check{x}}-q\right)-v_1^{(1)}\left((b_3-a_3)e^{\check{x}}-q\right)\right]}{v_1^{(3)}v_3^{(1)}-v_1^{(1)}v_3^{(3)}}$$

where

$$b_1 = \frac{\eta^L\left(\zeta_1+\frac{\tilde{\zeta}_2}{\delta^L}\right)+\Gamma^L\left(\zeta_1+\frac{\tilde{\zeta}_2}{\delta^H}\right)}{\Gamma^L\Gamma^H-\eta^H\eta^L}$$
$$b_3 = \frac{\eta^H\left(\zeta_1+\frac{\tilde{\zeta}_2}{\delta^H}\right)+\Gamma^H\left(\zeta_1+\frac{\tilde{\zeta}_2}{\delta^L}\right)}{\Gamma^L\Gamma^H-\eta^H\eta^L}$$

and $\widehat{x}, \check{x}$ denote the optimal investment thresholds in the H and L regimes respectively.

5.4 Proof of Lemma 3

Proof. Due to symmetry, I focus on $f^H(t,x)$. By Dynkin's formula,

$$\mathbb{E}_t\left[f^H(T,x)\right] = f^H(t,x)+\mathbb{E}_t\left[\int_t^T\left(\mathcal{D}(f^H(s,x))+\frac{\partial f^H(s,x)}{\partial s}+\eta^L\Big[f^L(t,x)-f^H(t,x)\Big]\right)\mathbf{d}s\right]$$

using the boundary condition $f^H(T,x)=0$ and the definition of $f^H(t,x)$,

$$\mathbb{E}_t\left[\int_t^T\left(\mathcal{D}(f^H(s,x))+\frac{\partial f^H(s,x)}{\partial s}+\eta^L\Big[f^L(t,x)-f^H(t,x)\Big]\right)\mathbf{d}s\right] = -\mathbb{E}_t\left[\int_0^T \mathbb{1}_{x_s\geq\widehat{x}^H}\mathbf{d}s\right]$$

Bringing the expectations operator inside the integral via Fubini's theorem, applying the Radon-Nikodym theorem and by Lebesgue differentiation, we have

$$\mathcal{D}(f^H(t,x))+\frac{\partial f^H(t,x)}{\partial t}+\eta^L\Big[f^L(t,x)-f^H(t,x)\Big] = -\mathbb{1}_{x>\widehat{x}^H}\quad\text{a.e.}$$

$\square$

5.5 *Proof of Proposition 4*

Starting with the system of PDEs:

$$\frac{\partial f^H(t,x)}{\partial t} + \left[\mu^H - \frac{1}{2}(\sigma^H)^2\right]\frac{\partial f^H(t,x)}{\partial x} + \frac{1}{2}(\sigma^H)^2\frac{\partial^2 f^H(t,x)}{\partial x^2} + \eta^L\Big[f^L(t,x) - f^H(t,x)\Big] = -\mathbb{1}_{x>\widehat{x}^H}$$
$$\frac{\partial f^L(t,x)}{\partial t} + \left[\mu^L - \frac{1}{2}(\sigma^L)^2\right]\frac{\partial f^L(t,x)}{\partial x} + \frac{1}{2}(\sigma^L)^2\frac{\partial^2 f^L(t,x)}{\partial x^2} + \eta^H\Big[f^H(t,x) - f^L(t,x)\Big] = -\mathbb{1}_{x>\widehat{x}^L}$$

I reduce the above system into a system of ODEs via the Laplace transform, $\overline{f}^H(\xi,x) = \mathscr{L}\{f^H\} \doteq \int_0^\infty e^{-\xi t} f^H(t,x)\mathbf{d}t$. The system can be rewritten:

$$\xi\overline{f}^H(\xi,x) + \left[\mu^H - \frac{1}{2}(\sigma^H)^2\right]\frac{\partial\overline{f}^H(\xi,x)}{\partial x} + \frac{1}{2}(\sigma^H)^2\frac{\partial^2\overline{f}^H(\xi,x)}{\partial x^2} + \eta^L\Big[\overline{f}^L(\xi,x) - \overline{f}^H(\xi,x)\Big] = -\frac{1}{\xi}\mathbb{1}_{x>\widehat{x}^H}$$
$$\xi\overline{f}^L(\xi,x) + \left[\mu^L - \frac{1}{2}(\sigma^L)^2\right]\frac{\partial\overline{f}^L(\xi,x)}{\partial x} + \frac{1}{2}(\sigma^L)^2\frac{\partial^2\overline{f}^L(\xi,x)}{\partial x^2} + \eta^H\Big[\overline{f}^H(\xi,x) - \overline{f}^L(\xi,x)\Big] = -\frac{1}{\xi}\mathbb{1}_{x>\widehat{x}^L}$$

As we have seen, whether $\widehat{x}^H > \widehat{x}^L$ or $\widehat{x}^H < \widehat{x}^L$ can depend on the values of μ^H and μ^L. I focus on more common case in which $\widehat{x}^H > \widehat{x}^L$, i.e., the no-action region expands in the high volatility regime. Then, the system of ODEs is split into three regions: $x < \widehat{x}^L$, $\widehat{x}^L < x < \widehat{x}^H$ and $x > \widehat{x}^H$.

As before, the general solution is given by:

$$\mathbf{F} = \sum_i^4 c_i\mathbf{v}^{(i)}e^{\lambda^{(i)}x}$$

where c_i are integration constants and $\mathbf{v}^{(i)}$ is the eigenvector associated with the ith eigenvalue $\lambda^{(i)}$. The eigenvalues are the solutions to the quartic equation

$$\left(\lambda^2 + \frac{\mu^H - .5(\sigma^H)^2}{.5(\sigma^H)^2}\lambda + \frac{\xi - \eta^L}{.5(\sigma^H)^2}\right)\left(\lambda^2 + \frac{\mu^L - .5(\sigma^L)^2}{.5(\sigma^L)^2}\lambda - \frac{\xi - \eta^H}{.5(\sigma^L)^2}\right) - \left(\frac{\eta^H}{.5(\sigma^H)^2}\right)\left(\frac{\eta^L}{.5(\sigma^L)^2}\right) = 0$$

which can be solved explicitly. Solving the ODEs piecewise and matching at $x = \widehat{x}^H$ and at $x = \widehat{x}^L$, we have the following solution:

$\underline{x < \widehat{x}^L :}$

$$\overline{f}^H(\xi, x) = c_1^H e^{\lambda^{(1)}(x-\widehat{x}^L)} + c_3^H e^{\lambda^{(3)}(x-\widehat{x}^L)}$$
$$\overline{f}^L(\xi, x) = c_1^L e^{\lambda^{(1)}(x-\widehat{x}^L)} + c_3^L e^{\lambda^{(3)}(x-\widehat{x}^L)}$$

$\underline{\widehat{x}^L < x < \widehat{x}^H :}$

$$\overline{f}^H(\xi, x) = \mathfrak{c}_1^H e^{\lambda^{(1)}(x-\widehat{x}^H)} + \mathfrak{c}_2^H e^{\lambda^{(2)}(x-\widehat{x}^L)} + \mathfrak{c}_3^H e^{\lambda^{(3)}(x-\widehat{x}^H)} + \mathfrak{c}_4^H e^{\lambda^{(4)}(x-\widehat{x}^L)} + \frac{\eta^L}{\xi^2(\xi - \eta^H - \eta^L)}$$
$$\overline{f}^L(\xi, x) = \mathfrak{c}_1^L e^{\lambda^{(1)}(x-\widehat{x}^H)} + \mathfrak{c}_2^L e^{\lambda^{(2)}(x-\widehat{x}^L)} + \mathfrak{c}_3^L e^{\lambda^{(3)}(x-\widehat{x}^H)} + \mathfrak{c}_4^L e^{\lambda^{(4)}(x-\widehat{x}^L)} - \frac{\xi - \eta^L}{\xi^2(\xi - \eta^H - \eta^L)}$$

$\underline{\widehat{x}^H < x :}$

$$\overline{f}^H(\xi, x) = \mathcal{c}_2^H e^{\lambda^{(2)}(x-\widehat{x}^H)} + \mathcal{c}_4^H e^{\lambda^{(4)}(x-\widehat{x}^H)} - \frac{1}{\xi^2}$$
$$\overline{f}^L(\xi, x) = \mathcal{c}_2^L e^{\lambda^{(2)}(x-\widehat{x}^H)} + \mathcal{c}_4^L e^{\lambda^{(4)}(x-\widehat{x}^H)} - \frac{1}{\xi^2}$$

The 16 integration constants have explicit forms and can be found by imposing matching at $x = \widehat{x}^H$ and at $x = \widehat{x}^L$, along with boundary conditions at $x \to -\infty$ and $x \to \infty$ (for example, $\overline{f}^L(\xi, -\infty) = 0$). Moreover, $\lambda^{(1)}, \lambda^{(3)}$ are the positive eigenvalues to the characteristic equation while $\lambda^{(2)}, \lambda^{(4)}$ are the negative ones.

With explicit solutions for $\overline{f}^H(\xi, x)$ and $\overline{f}^L(\xi, x)$, all that remains is to invert them. Remarkably, the poles lie at $\xi = 0$ and $\xi = \eta^H + \eta^L$. Hence, the expected occupation time can be found by evaluating the behavior of $\overline{f}$ at these two points, which can easily be done numerically.

6 Appendix: Supplementary Figures

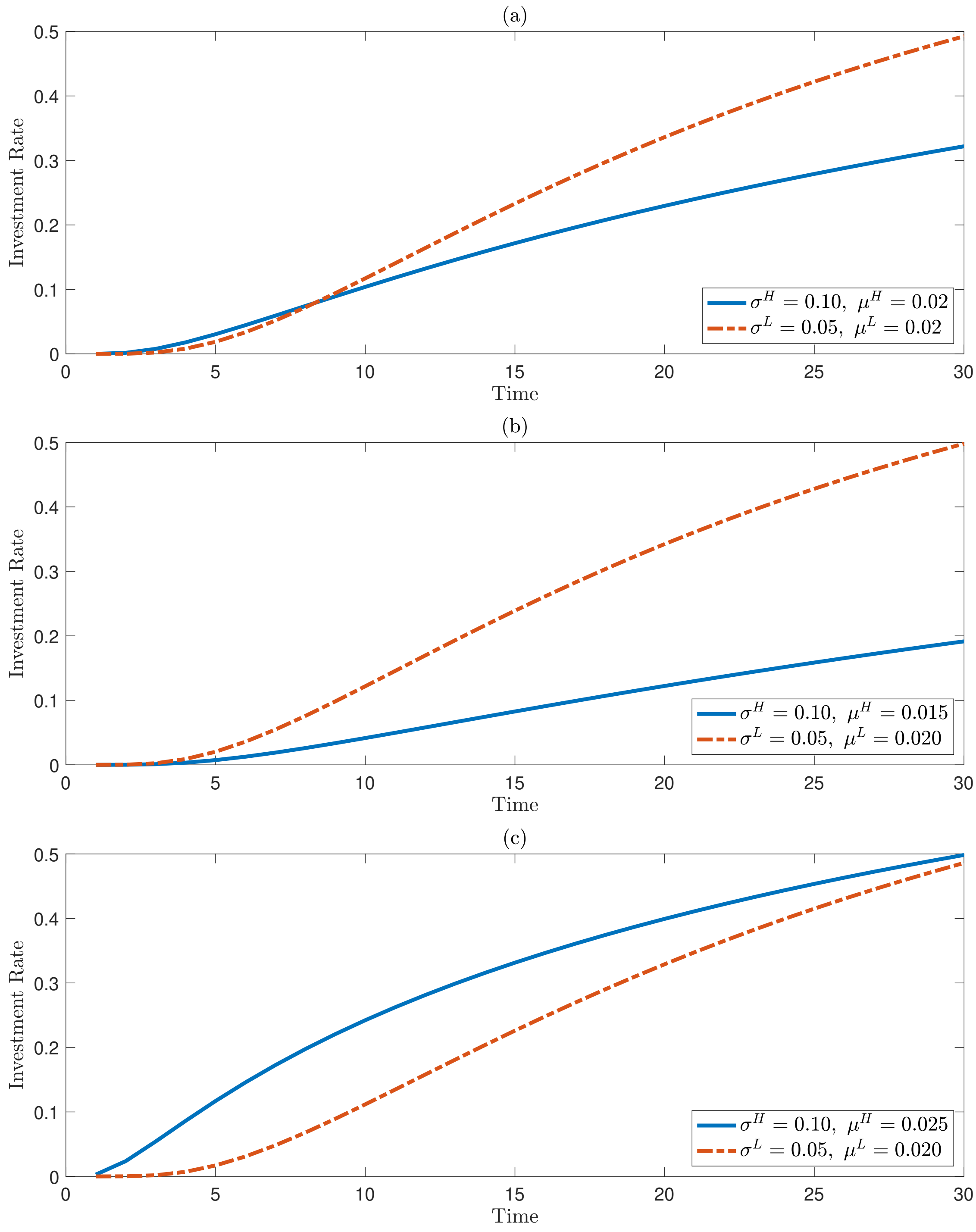
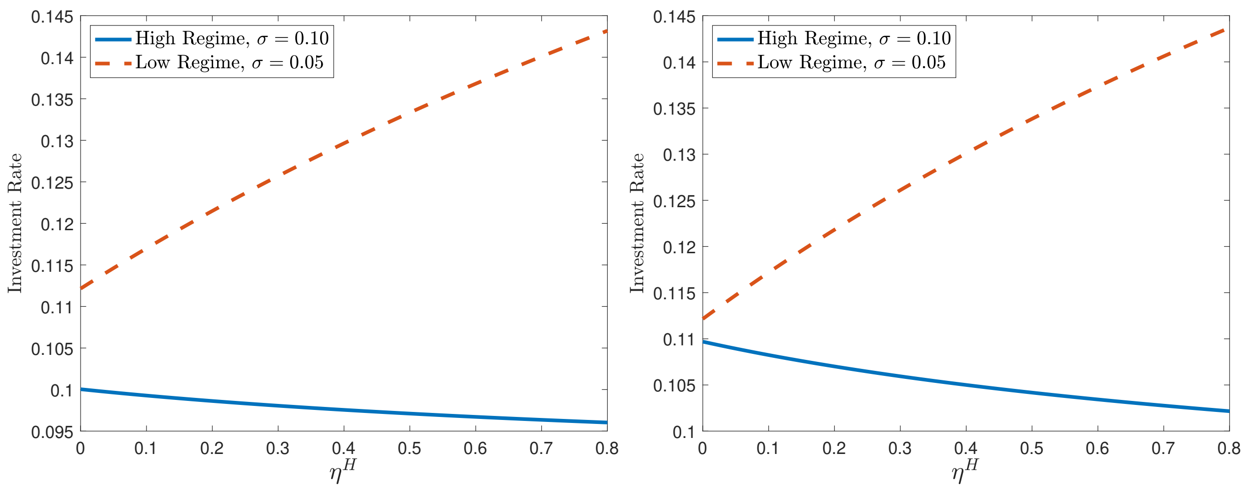


Figure IA.9: **Investment rate with stochastic volatility as a function of the time span, fixing the initial point at $x_0 = 0$.** This figure studies the following cases: $\mu^H = \mu^L,\ \mu^H > \mu^L$, and $\mu^H < \mu^L$.



(a) $\eta^L = 0.20$

(b) $\eta^L = 0.40$

Figure IA.10: **Investment rate with stochastic volatility as a function of the arrival rate of the high regime η^H.**